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Hybrid pair of mappings and common fixed point theorems in ordered cone metric spaces over Banach algebras

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Abstract

The purpose of this paper is to establish some coincidence and common fixed point results of hybrid pair of a single-valued and a set-valued mapping on an ordered cone metric spaces over Banach algebras. Our results extend, generalize, and unify several known fixed point results on cone metric spaces equipped with a partial order. Some examples are presented which verify the significance of the results proved herein.

Keywords: *Cone metric space; point of coincidence; common fixed point; partial order.*

1. Introduction

Let (X,d) be a metric space. Denote by $CB(X)$, the set of all closed and bounded subsets of the space X. Pompeiu and then Hausdorff introduced a function $d_H: CB(X)\times CB(X)\to\mathbb{R}$ such that the pair $(CB(X), d\_H)$ forms a metric space itself (for details, see, [5,6,12–16]). The function d_H is called the Pompeiu-Hausdorff metric on $CB(X)$ and it is defined as follows:

$$d_H(A,B)=\max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}$$

for all $A,B\in CB(X)$.

Nadler [38] generalized the famous Banach contraction principle for the mappings with domain X and the codomain $CB(X)$ and proved the following theorem:

Theorem 1.1 (Nadler [38]). Let (X,d) be a complete metric space and $T: X\to CB(X)$ be a set valued mapping. If there exists $\lambda\in(0,1)$ such that

$$d_H(T(x),T(y))\le \lambda d(x,y), \forall\, x,y\in X.$$

Then T has a fixed point in X.

Huang and Zhang [27] defined cone metric spaces and convergent and Cauchy sequences in cone metric spaces in terms of interior points of the underlying cone. Some basic versions of the fixed point theorems in cone metric spaces can be found in [27].

The concept of commutativity of two mappings and several of its weaker forms, like compatibility, weak compatibility, R-weak commutativity etc. have been extended in their corresponding set-valued forms. For the pair of a single-valued and set-valued mappings (hybrid pair), similar concepts have been introduced and studied by several authors, see, e.g., [2,3,30,31,33,37].

Wardowski [10] for a cone metric space (M,d) and for the family $\mathcal{A}$ of subsets of M established a new cone metric $H:\mathcal{A}\times\mathcal{A}\to E$, where E is an ordered Banach space. He introduced the concept of set-valued contraction of Nadler type in cone metric spaces and prove a fixed point theorem (see also, [9,11]).

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2. Preliminaries

In this section, we recall some well-known definitions which will be needed in the sequel and can be found in [10,22,23,26,34,40].

Definition 2.1. Let $\mathcal{B}$ be a real Banach algebra, i.e., $\mathcal{B}$ is a real Banach space with aproduct that satisfiesis a real Banach space with aproduct that satisfies:

1. $x(yz) = (xy)z$; 2. $x(y+z) = xy + xz$;3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;4. $\|xy\| \leq \|x\|\|y\|$,for all $x, y, z \in \mathcal{B}, \alpha \in \mathbb{R}$.

The Banach algebra $\mathcal{B}$ is said to be unital if there is an element $e \in \mathcal{B}$ such that$ex = xe = x$ for all $x \in \mathcal{B}$. The element e is called the unit. An $x \in \mathcal{B}$is said to be invertible if there is a $y \in \mathcal{B}$ such that $xy = yx = e$. The inverse of x, if it exists, is unique and will be denoted by x^{-1}. For more details, see [40].

Proposition 2.2 [(40)]. Let $\mathcal{B}$ be a Banach algebra with unit eand $x \in \mathcal{B}$. If the spectral radius $\rho(x)$ of x and $\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} < 1$ then $e - x$ is invertible and $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Let $\mathcal{B}$ be a unital Banach algebra. A non-empty closed set $P \subset \mathcal{B}$ is said to be a cone (see [7,8]) if $e \in P$ and: (1) $\alpha P + \beta P, P^2 \subset P$ for all $\alpha, \beta \geq 0$, (2) $P \cap (-P) = \{\theta\}$, where θ is the zero vector of $\mathcal{B}$. Given a cone P $\subset \mathcal{B}$ one can define a partial order $\preccurlyeq$ on $\mathcal{B}$ by $x \preccurlyeq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^{\circ}$, where P° denotes the interior of P.

Lemma 2.3 ([19]). Let $P \subset \mathcal{B}$ be a solid cone and $a, b, c \in P$.

(a). If $a \preccurlyeq b$and $b \ll c$, then $a \ll c$.

(b). If $a \ll b$and $b \ll c$, then $a \ll c$.

(c). If $\theta \preccurlyeq u \ll c$ for each $c \in P^{\circ}$, then $u = \theta$.

Lemma 2.4 ([34]). Let $\mathcal{B}$ be a Banach algebra with a unit e, P be a cone in $\mathcal{B}$ and$a, b, c \in P$.

(i) If $\rho(a) < 1$, then $\rho(a^m) \leq \rho(a) < 1$ for each $m \in N$.

(ii) If $\rho(a) < 1$ and $b \preccurlyeq ac$, then $b \preccurlyeq c$.

Remark 2.5 ([39]). If $\rho(a) < 1$ then $\|a^n\| \to 0$ as $n \to \infty$.

Lemma 2.6 ([21]). Let P be a cone and $k \in P$with $\rho(k) < 1$. Then, for every $c \in P^{\circ}$there exists $n_0 \in \mathbb{N}$ such that $k^n \ll c$ for all $n > n_0$.

Henceforth, we will assume that the real Banach algebra B is unital and that the cone $P \subset \mathcal{B}$ is a solid cone, i.e.,$P^{\circ} \neq \emptyset$.

Definition 2.7 ([22,23,27]). Let X be a non-empty set. Suppose that the mapping $d: X \times X \to \mathcal{B}$ satisfies:

1) $\theta \preccurlyeq d(x,y)$for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$.
2) $d(x,y) = d(y,x)$for all $x, y \in X$.
3) $d(x,y) \preccurlyeq d(x,z) + d(z,y)$for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over the Banach algebra$\mathcal{B}$.

Definition 2.8 ([22,23,27]). Let (X,d) be a cone metric space over the Banach algebra $\mathcal{B}$, $x \in X$ and $\{x_n\}$ be a sequence in$X$. Then:

(i). $\{x_n\}$is said to be convergent to x if for every $c \in \mathcal{B}$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or$x_n \to x$ as $n \to \infty$.

(ii). $\{x_n\}$ is called a Cauchy sequence if for every $c \in \mathcal{B}$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.

(iii). $(X,d)$is called complete if every Cauchy sequence in X converges to some point in X.

(iv). A mapping $f: X \to X$ is called continuous if every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to \infty$ we have $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.9 ([10,34]). Let (X,d) be a cone metric space over a Banach algebra $\mathcal{B}$ and let $\mathcal{A}$ be a collection of nonempty subsets of X. A map $H: \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is called a H −cone metric with respect to d if for any $A_1, A_2 \in \mathcal{A}$ the following conditions hold:

(H1) $H(A_1, A_2) = 0 \Rightarrow A_1 = A_2$;

(H2) $H(A_1, A_2) = H(A_2, A_1)$;

(H3) $\forall c \in E, \theta \ll c \forall x \in A_1, \exists y \in A_2$:

$$d(x,y) \preccurlyeq H(A_1, A_2) + c;$$

(H4) One of the following is satisfied:

(i) $\forall c \in E, \theta \ll c \exists x \in A_1, \forall y \in A_2 H(A_1, A_2) \preccurlyeq d(x,y) + c$;

(ii) $\forall c \in E, \theta \ll c \exists x \in A_2, \forall y \in A_1 H(A_1, A_2) \preccurlyeq d(x,y) + c$;

It is obvious that each $H-$cone metric depends on the choice of the collection $\mathcal{A}$. For examples of $H-$cone metrics on cone metric spaces, see [10].

Remark 2.10. Let (X,d) be a cone metric space over a Banach algebra $\mathcal{B}$ and let $\mathcal{A}$ bea collection of nonempty subsets of$X$. If $\mathcal{A}$ contains two singleton sets $\{x\}$ and $\{y\}, x, y \in X$, then it follows from the definition of H -cone metric that $H(\{x\},\{y\}) = d(x,y)$.

Definition 2.11. Let X be a nonempty set and $T: X \to 2^X$and $g: X \to X$ be twomappings. Then, the pair (T,g) is called a hybrid pair. If $x, y \in X$, then x is called acoincidence point of the pair (T,f) and y is called the corresponding point of coincidenceif

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(see also, [17,18,35]) if $y = gx \in Tx$. The set of all coincidence point of the pair (T, f) is denoted by $CP(T, f)$.

Remark 2.12. Note that, if $g = I_X$, the identity mapping of X, then a pointof coincidence of the hybrid pair (T, g) reduces into the fixed point of the set-valuedmapping$T$. Therefore, the fixed point of a set-valued mapping is a particular case of the point of coincidence of the hybrid pair.

Now we can state our main results.

3. Main Results

In this section, we introduce some new notions and prove some common fixed point results in cone metric spaces over Banach algebras and equipped with a partial order.

Definition 3.1. Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) is a cone metric space over Banach algebra$\mathcal{B}$. Suppose, $\mathcal{A}$be a nonempty collection of nonempty closed subsets of X. Then, a mapping $T: X \to X$ is called a generalized ordered Nadler contraction with contractive vector a, if for all $x, y \in X$ with $x \sqsubseteq y$:

(O1) there exists $a \in P$ such that $\rho(a) < 1$ and $H(Tx, Ty) \preccurlyeq ad(x, y)$;

(O2) if $u \in Tx$, $v \in Ty$ are such that $d(u, v) \preccurlyeq d(x, y)$, then $u \sqsubseteq v$.

Definition 3.2. Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a cone metric space over Banach algebra$\mathcal{B}$. Suppose, $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X and $g: X \to X$ be a mapping. Then, a mapping $T: X \to \mathcal{A}$ is called a generalized g-ordered Nadler contraction with contractive vector a, if for all $x, y \in X$ with $gx \sqsubseteq gy$:

(O g 1) there exists $a \in P$ such that $\rho(a) < 1$ and $H(Tx, Ty) \preccurlyeq ad(gx, gy)$;

(O g 2) if $gu \in Tx$, $gv \in Ty$ are such that $d(gu, gv) \preccurlyeq d(gx, gy)$, then $gu \sqsubseteq gv$.

Remark 3.3. It is easy to see that the generalized ordered Nadler contraction with some contractive vector is a particular case of generalized It is easy to see that the generalized ordered Nadler contraction with some contractive vector is a particular case of generalized g -ordered Nadler contractions. In particular, every generalized ordered Nadler contraction with contractive vector -ordered Nadler contractions. In particular, every generalized ordered Nadler contraction with contractive vector a is a generalized is a generalized I_X -ordered Nadler contraction with contractive vector a, where I_X is the identity mapping of set is the identity mapping of set X. On the other hand, the converse is not true in general (see, Example 3.7 of this paper).

Let $(X, \sqsubseteq)$ be a partially ordered set, $x \in X$ and $A \subseteq X$. Then we write $x \sqsubseteq A$ if there exists $y \in A$ such that $x \sqsubseteq y$.

Theorem 3.4. Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a cone metric space over a Banach algebra $\mathcal{B}$. Suppose, $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X, $g: X \to X$ be a mapping and $T: X \to \mathcal{A}$ be a generalized g-ordered Nadler contraction with contractive vector a. Suppose, $Tx \subseteq g(X)$ for all $x \in X$, $g(X)$ is complete and the following conditions hold:

(I) there exists $x_0 \in X$ such that $gx_0 \sqsubseteq Tx_0$;

(II) for any sequence $\{gx_n\} \in X$, if $gx_n \to gx$ and $gx_n \in Tx_{n-1}, gx_{n-1} \sqsubseteq gx_n$ for all $n \in \mathbb{N}$, then there is a subsequence $\{gx_{n_j}\}$ such that $gx_{n_j} \sqsubseteq gx$ for all $j \in \mathbb{N}$.

Then, the hybrid pair (T, g) has a coincidence point, and there exists a non-decreasing sequence $\{z_n\} \in X$ such that $z_{n-1} \sqsubseteq z_n$, $z_n \in Tz_{n-1}$ for all $n \in \mathbb{N}$ and it converges to the coincidence point of the pair (T, g).

Proof. Suppose, $x_0 \in X$ is such that $gx_0 \sqsubseteq Tx_0$. Then, there exists $z_1 \in Tx_0$ such that $gx_0 \sqsubseteq z_1$. As, $Tx_0 \subseteq g(X)$ and $z_1 \in Tx_0$, there exists $x_1 \in X$ such that $z_1 = gx_1$. Let $z_0 = gx_0$, then we have $z_0 \sqsubseteq z_1$, i.e., $gx_0 \sqsubseteq gx_1$. Since $z_1 \in Tx_0$, by definition of H-cone metric there exists $z_2 \in Tx_1$ such that

$$d(z_1, z_2) \preccurlyeq H(Tx_0, Tx_1) + c_1$$

where $c_1 \in P^\circ$ is chosen so that $\rho(c_1) < 1$. As, $gx_0 \sqsubseteq gx_1$ and T is a generalized g-ordered Nadler contraction with contractive vector a, it follows from the above inequality that

$$d(z_1, z_2) \preccurlyeq ad(gx_0, gx_1) + c_1. \qquad (1)$$

Letting $c_1 \to \theta$ it follows from the above inequality that

$$\mathrm{d}(\mathrm{z}_1, \mathrm{z}_2) \preccurlyeq \mathrm{ad}(\mathrm{gx}_0, \mathrm{gx}_1) \preccurlyeq \mathrm{d}(\mathrm{gx}_0, \mathrm{gx}_1) = \mathrm{d}(\mathrm{z}_0, \mathrm{z}_1).$$

As, $z_2 \in Tx_1$ and $Tx_1 \subseteq g(X)$, so, there exists $x_2 \in X$ such that $z_2 = gx_2$. Again, since $gx_0 \sqsubseteq gx_1$, $z_1 = gx_1 \in Tx_0, z_2 = gx_2 \in Tx_1$ and $d(z_1, z_2) = d(gx_1, gx_2) \preccurlyeq d(gx_0, gx_1)$, it follows from (O$g$2) that $z_1 \sqsubseteq z_2$, i.e., $gx_1 \sqsubseteq gx_2$.

Repeating similar arguments, we obtain a sequence $\{z_n\} = \{gx_n\}$ such that $z_{n-1} \sqsubseteq z_n$, i.e., $gx_{n-1} \sqsubseteq gx_n$, $z_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and the following inequality is satisfied:

$$d(z_n, z_{n+1}) \preccurlyeq ad(gx_{n-1}, gx_n) + c_n = ad(z_{n-1}, z_n) + c_n$$

where $c_n \in P^\circ$ is chosen so that $\rho(c_n) < 1$.

Successive use of the above inequality yields

$$d(z_n, z_{n+1}) \preccurlyeq a^n d(z_0, z_1) + \sum_{i=0}^{n-1} a^i c_{n-i}.$$

Note that, since $\rho(a) < 1$, by Lemma 2.4, we have $\rho(a^n) < 1$ for all $n \in \mathbb{N}$. Therefore, we can choose $c_n = a^{2n}$ for all $n \in \mathbb{N}$, also, we have $(e - a)^{-1} = \sum_{i=0}^{\infty} a^i$, and so, it follows from the above inequality that

$$d(z_n, z_{n+1}) \preccurlyeq a^n d(z_0, z_1) + a^n (e-a)^{-1}. \quad (3)$$

Then for every $n, m \in \mathbb{N}$ with $m > n$ we have

$$d(z_n, z_m) \preccurlyeq \sum_{j=n}^{m-1} d(z_j, z_{j+1})$$
$$\preccurlyeq a^n (e-a)^{-1} d(z_0, z_1) + a^n (e-a)^{-2}.$$

Since, $\rho(a) < 1$ by Remark 2.5 we have $\|a^n\| \to 0$, i.e., $a^n \to \theta$ as $n \to \infty$. Hence, $a^n(e-$ $a-1dz0,z1+ane-a-2\to\theta$ as $n\to\infty$. Now by part (a) and (d) of Lemma 2.3 it follows that, for given $c \in P$, $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(z_n, z_m) \ll c$ for all $n, m > n_0$. Thus, $\{z_n\} = \{gx_n\}$ is a Cauchy sequence. By completeness of $g(X)$, there exists $x^* \in X$ such that $z_n \to gx^* = y^*$ (say) as $n \to \infty$.

We shall show that x^* is a coincidence point of T and g.

By (II) there exists a subsequence $\{z_{n_j}\} = \{gx_{n_j}\}$ of $\{z_n\}$ such that $gz_{n_j} \sqsubseteq gx^*$ for all $j \in \mathbb{N}$. Then, since $z_n = gx_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$, by definition of H-cone metric, for each $j \in \mathbb{N}$ there exists $y_j \in Tx^*$ such that

$$d\left(z_{n_j+1}, y_j\right) \preccurlyeq H\left(Tx_{n_j}, Tx^*\right) + c_j$$

where $c_j \in P^\circ$ is such that $c_j \to \theta$ as $j \to \infty$. Since $gz_{n_j} \sqsubseteq gx^*$ for all $j \in \mathbb{N}$ and T is a generalized g-ordered Nadler contraction with contractive vector a, it follows from the above inequality that

$$d\left(z_{n_j+1}, y_j\right) \preccurlyeq ad\left(gx_{n_j}, gx^*\right) + c_j.$$

Using the above inequality we obtain

$$d(y_j, gx^*) \preccurlyeq d\left(y_j, z_{n_j+1}\right) + d\left(z_{n_j+1}, gx^*\right)$$
$$\preccurlyeq ad\left(gx_{n_j}, gx^*\right) + c_j + d\left(z_{n_j+1}, gx^*\right).$$

Since $z_j \to gx^*$ and $c_j \to \theta$ as $j \to \infty$, for any $c \in P^\circ$, there exists $n_1 \in \mathbb{N}$ such that $d\left(z_{n_j+1}, gx^*\right) + ad\left(z_{n_j}, gx^*\right) + c_j \ll c$ for all $j > n_1$. Therefore, using part (a) of Lemma 2.3 in the above inequality gives

$$d(y_j, gx^*) \ll c \text{ for all } j > n_1.$$

It shows that $y_j \to gx^*$ as $j \to \infty$. Since $y_j \in Tx^*$ for all $j \in \mathbb{N}$ and Tx^* is closed we must have $gx^* \in Tx^*$. Thus, x^* is a coincidence point of T and g.

The following corollary is an ordered version of the result of Wardowski [10].

Corollary 3.5. Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a Banach algebra $\mathcal{B}$. Suppose, $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X and $T: X \to \mathcal{A}$ be a generalized-ordered Nadler contraction with contractive vector a and there exists $x_0 \in X$ such that:

(I) there exists $x_0 \in X$ such that $gx_0 \sqsubseteq Tx_0$;

(II) for any sequence $\{x_n\} \in X$, if $x_n \to x$ and $x_n \in Tx_{n-1}$, $x_{n-1} \sqsubseteq x_n$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \sqsubseteq x$ for all $j \in \mathbb{N}$.

Then, the T has a fixed point in X, and there exists a non-decreasing sequence $\{z_n\} \in X$ such that $z_{n-1} \sqsubseteq z_n$, $z_n \in Tz_{n-1}$ for all $n \in \mathbb{N}$ and it converges to the fixed point of T.

Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) is a cone metric space over Banach algebra $\mathcal{B}$. Suppose, $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X and $T: X \to \mathcal{A}$ be a mapping. Then T is called ordered closed, if for sequences $\{x_n\}, \{y_n\}$ in X such that $x_n \sqsubseteq y_n$, $x_n \in Ty_n$ for all $n \in \mathbb{N}$ and $x_n \to x$, $y_n \to y$ as $n \to \infty$, we have $x \in Ty$. If $g: X \to X$ is a mapping, then the mapping T is called ordered g-closed, if for sequences $\{gx_n\}, \{y_n\}$ in X such that $gx_n \sqsubseteq gy_n$, $gx_n \in Ty_n$ for all $n \in \mathcal{B}$ and $gx_n \to gx$, $gy_n \to gy$ as $n \to \infty$, we have $gx \in Ty$.

A subset S of X is called well-ordered if $x \sqsubseteq y$ for all $x, y \in S$. The set S is called weakly well-ordered if there exist at least one pair (x, y) in $S \times S$ such that $x \sqsubseteq y$. The set S is called g-weakly well-ordered if there exist at least one pair (gx, gy) in $g(S) \times g(S)$ such that $gx \sqsubseteq gy$ or $gy \sqsubseteq gx$.

In the next theorem we replace the condition (II) of Theorem 3.4 by ordered g-closedness of T.

Theorem 3.6. Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a cone metric space over a Banach algebra $\mathcal{B}$. Suppose, $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X, $g: X \to X$ be a mapping and $T: X \to \mathcal{A}$ be a generalized g-ordered Nadler contraction with contractive vector a. Suppose, $Tx \subseteq g(X)$ for all $x \in X$, $g(X)$ is complete and the following conditions hold:

(I) there exists $x_0 \in X$ such that $gx_0 \sqsubseteq Tx_0$;

(II) T is ordered g-closed.

Then, the hybrid pair (T, g) has a coincidence point, and there exists a non-decreasing sequence $\{z_n\} \in X$ such that $z_{n-1} \sqsubseteq z_n$, $z_n \in Tz_{n-1}$ for all $n \in \mathbb{N}$ and it converges to the coincidence point of the pair (T, g).

Proof. Following the same lines of proof of Theorem 3.4 we obtain the sequence $\{z_n\}$ such that $\{z_n\} = \{gx_n\}$ such that $z_{n-1} \sqsubseteq z_n$, i.e., $gx_{n-1} \sqsubseteq gx_n$, $z_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and $z_n \to gx^* = y^*$ (say) as $n \to \infty$ for some $x^* \in X$. Since T is ordered g-closed, $gx_{n-1} \sqsubseteq gx_n$, $gx_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and $gx_n \to gx^* = y^*$ for some $x^* \in X$, so, we have $gx^* \in Tx^*$. Thus, Thus, x^* is a coincidence point of T and g.

Example 3.7. Let $X = [0,1]$, $\mathcal{B} = \mathbb{R}^2$ be a Banach space with the standard norm, coordinate-wise multiplication and unite $e = (1,1)$ and $P = \{(x, y): x, y \geq 0\}$ be a solid cone and let $d: X \times X \to E$ be of the form $d(x, y) = (|x-y|, 1/2|x-y|)$. Then the pair (X, d) is a complete cone metric space over Banach algebra $\mathcal{B}$. Let $\mathcal{A}$ be a family of subsets of X of the form $A = \{[0,x]: x \in X\} \cup \{\{x\}: x \in X\}$. We define an H-cone metric $H: A \times A \to E$ with respect to d by the formulae:

$$H(A,B) = \begin{cases} \left(1,\frac{1}{2}\right)|x-y| & if\ A = [0,x], B = [0,y] \\ \left(1,\frac{1}{2}\right)|x-y| & if\ A = \{x\}, B = \{y\} \\ \left(\max\{y,|x-y|\}, \frac{1}{2}\max\{y,|y-x|\}\right) & if A = [0,x], B = \{y\} \\ \left(\max\{x,|x-y|\}, \frac{1}{2}\max\{x,|y-x|\}\right) & if B = [0,y], B = \{x\} \end{cases}.$$

Define the mapping $T: X \to \mathcal{A}$ as follows:

$$Tx = \begin{cases} \beta\ if\ x \in [0,\alpha) \\ [0.x^2]\ if\ x \in (\alpha,1] \end{cases}$$

$\alpha \in [0,1)$ and $\beta \in [0,\alpha)$ are two fixed number. Suppose, $\sqsubseteq$ is a partial order on X and defined as follows:

$\sqsubseteq = \{(x,y): y \leq x, x,y \in [0,\alpha)\} \cup \{(x,x): x \in X\}$.

Then, all the conditions of Theorem 3.6 are satisfied with $g = I_X$ and we can conclude the existence of fixed point of T. On the other hand, it is easy to see that T is not a contraction in the sense of Wardowski [10], therefore the result of Wardowski [10] cannot be applied here.

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