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A study on projective recurrent and symmetric tensor in almost kaehlerian spaces

Trishna Devi[1], U.S.Negi[2] and Mahender Singh Poonia[3]

[1] Research Scholar, Department of Mathematics, Shri Jagdish Prasad Jhabarmal Tibrewala University, Vidyanagari, Jhunjhunu, Rajasthan – 333001, India

[2]Department of Mathematics, H.N.B. Garhwal (A Central) University, S.R.T. Campus Badshahi Thaul, Tehri Garhwal – 249 199, Uttarakhand, India.

[3]Associate Professor, Department of Mathematics, Shri Jagdish Prasad Jhabarmal Tibrewala University, Vidyanagari, Jhunjhunu, Rajasthan – 333001, India

Abstract
In this paper, we have defined and studied on projective recurrent and symmetric tensor in almost Kaehlerian spaces and several theorems have been investigated. The condition that an almost Kaehlerian space be a projective recurrent space of the first order, second order and first kind, second order and Second kind are established. Also, The condition that an almost Kaehlerian space be a projective symmetric space of the first order and first kind, second order and second kind, etc are proved.

Key words: *Kaehlerian, Projective, Recurrent, Symmetric, and Space*. **2010 MSC:** 32C15, 46A13, 46M40, 53B35, 53C55.

1. Introduction

An almost Kaehler space is first of all an almost complex space, that is, a 2n-dimensional space with an almost complex structure F^h_i :

$$F^i_j F^h_i = -\delta^h_j, \tag{1.1}$$

and always admits a positive definite Riemannian metric tensor g_{ji} satisfying:

$$F^a_j F^b_i g_{ab} = g_{ji}, \tag{1.2}$$

from which

$$F_{ji} = -F_{ij}, \tag{1.3}$$

where

$$F_{ji} \overset{def}{=} F^a_j g_{ai} \tag{1.4}$$

and finally has the property that the differential form

$$F_{ji}\, d\xi^j \wedge d\xi^i \text{ is closed, that is,}$$

$$F_{jih} \overset{def}{=} \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0 \tag{1.5}$$

and the identity

$$F^i_j F_i = \tfrac{1}{2} F_{jih} F^{ih}, \tag{1.6}$$

where

$$F_i = -\nabla_j F^h_i \tag{1.7}$$

and ∇ denotes the operation of covariant differentiation with respect to the Riemannian connection $\{{}_j{}^i{}_k\}$.

If in an almost Kaehler space, the Nijenhuis tensor satisfies the condition

$$N_{jih} + N_{jhi} = 0,$$

then we deduce from it $G_{jih} = 0$, i.e.

$$\nabla_j F^h_i + \nabla_i F^h_j = 0 \tag{1.8}$$

and the space is an almost Tachibana space. Thus, we have

$$3\nabla_j F_{ih} = F_{jih} = 0.$$

Consequently, the space is a Kaehler space i.e. an almost Kaehler space is a Kaehler space, if and only if the Nijenhuis tensor equation is satisfied.

A Contravariant almost analytic vector field is defined as a vector field v^i, satisfying (Tachibana (1959)):

$$£_v F^h_i \equiv v^j \partial_j F^h_i - F^j_i \partial_j v^h + F^h_j \partial_i v^j = 0, \quad (1.9)$$

where $£_v$ stands for the Lie-derivative with respect to v^i.

2. Projective recurrent tensor in almost kaehlerian spaces

The holomorphic projective curvature tensor, P^h_{ijk}, is given by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}(R_{ik}\,\delta^h_j - R_{jk}\,\delta^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2S_{ij} F^h_k),$$

or equivalently

$$P_{ijkm} = R_{ijkm} + \frac{1}{n+2}(R_{ik}\, g_{jm} - R_{jk}\, g_{im} + S_{ik} F_{jm} - S_{jk} F_{im} + 2S_{ij} F_{km}),$$

Where R^h_{ijk} and $R_{ij} = R^a_{ija}$ are the Riemannian curvature and Ricci-tensor respectively. The tensor which is defined by

$$S_{ij} = R^r_i\, g_{rj}\ , \text{ Satisfies } S_{ij} = - S_{ji}.$$

We know that a space for which we have at every point

$$\nabla_a P_{ijkm} - \lambda_a P_{ijkm} = 0, \quad (2.1)$$

is called an almost Kaehlerian projective recurrent space.

DEFINITION (2.1): An almost Kaehler space, for which at every point, we have

$$\nabla_a (F^i_h P_{ijkm}) - \lambda_a F^i_h P_{ijkm} = 0, \quad (2.2)$$

will be called an almost Kaehlerian projective recurrent space of the first order and first kind.

DEFINITION (2.2): An almost Kaehler space, for which at every point, we have

$$\nabla_a (F^i_h F^j_t P_{ijkm}) - \lambda_a F^i_h F^j_t P_{ijkm} = 0, \quad (2.3)$$

will be called an almost Kaehlerian projective recurrent space of the second order and first kind.

DEFINITION (2.3): An almost Kaehler space, for which at every point, we have

$$\nabla_a (F^i_h F^k_s P_{ijkm}) - \lambda_a F^i_h F^k_s P_{ijkm} = 0, \quad (2.4)$$

will be called an almost Kaehlerian projective recurrent space of the second order and second kind.

DEFINITION (2.4): An almost Kaehler space, for which at every point, we have

$$\nabla_a (F^i_h F^j_t F^k_s P_{ijkm}) - \lambda_a F^i_h F^j_t F^k_s P_{ijkm} = 0, \quad (2.5)$$

will be called an almost Kaehlerian projective recurrent space of the third order and first kind.

DEFINITION (2.5): An almost Kaehler space, for which at every point, we have

$$\nabla_a (F^i_h F^j_t F^k_s F^m_n P_{ijkm}) - \lambda_a F^i_h F^j_t F^k_s F^m_n P_{ijkm} = 0, \quad (2.6)$$

will be called an almost Kaehlerian projective recurrent space of the fourth order and first kind.

We, now, have the following:

THEOREM (2.1): The condition that an almost Kaehlerian space be a projective recurrent space of the first order is

$$\nabla_a P_{rjkm} - \lambda_a P_{rjkm} = F^i_a F^h_r (\nabla_h P_{ijkm} - \lambda_h P_{ijkm}). \quad (2.7)$$

PROOF: Equation (2.2) is equivalent to

$$(\nabla_a F^i_h) P_{ijkm} + F^i_h \nabla_a P_{ijkm} - \lambda_a F^i_h P_{ijkm} = 0. \quad (2.8)$$

Interchanging the indices a and h in equation (2,8) and adding the result thus obtained in the above equation, we get after using (1.8),

$$(\nabla_a P_{ijkm} - \lambda_a P_{ijkm}) F^i_h + (\nabla_h P_{ijkm} - \lambda_h P_{ijkm}) F^i_a = 0, \quad (2.9)$$

Transvecting the above equation with F^h_r and using (1.1) we get the required condition (2.7).

THEOREM (2.2): The condition that an almost Kaehlerian space be a projective recurrent space of the second order and first kind is

$$\nabla_a (F^j_t P_{rjkm}) - \lambda_a F^j_t P_{rjkm} = [\nabla_h (F^j_t P_{ijkm} - \lambda_h F^j_t P_{ijkm})]F^i_a F^h_r \quad (2.10)$$

PROOF: Equation (2.3) is equivalent to

$$(\nabla_a F^i_h) F^j_t P_{ijkm} + F^i_h \nabla_a (F^j_t P_{ijkm}) - \lambda_a F^i_h F^j_t P_{ijkm} = 0. \quad (2.11)$$

Interchanging the indices a and h in the above equation and adding the result thus obtained in (2.11), we get after using (1.8),

$$F^i_h [\nabla_a (F^j_t P_{ijkm}) - \lambda_a F^j_t P_{ijkm}] + F^i_a [\nabla_h (F^j_t P_{ijkm}) - \lambda_h F^j_t P_{ijkm}] = 0. \quad (2.12)$$

Transvecting the above equation by F^h_r and using (1.1) we get the required condition (2.10).

THEOREM (2.3): The condition that an almost Kaehlerian space be a projective recurrent space of the second order and second kind is

$$\nabla_a (F^k_s P_{rjkm}) - \lambda_a F^k_s P_{rjkm} = [\nabla_h (F^k_s P_{ijkm}) - \lambda_h F^k_s P_{ijkm}] F^i_a F^h_r. \quad (2.13)$$

The proof is similar to the proof of theorem (2.2).

THEOREM (2.4): The conditions that an almost Kaehlerian space be a projective recurrent space of the third and fourth order are

$$\nabla_a (F^j_t F^k_s P_{rjkm}) - \lambda_a F^j_t F^k_s P_{rjkm} = [\nabla_h (F^j_t F^k_s P_{ijkm}) - \lambda_h F^j_t F^k_s P_{ijkm}] F^i_a F^h_r. \quad (2.14)$$

and

$$\nabla_a (F^j_t F^k_s F^m_n P_{rjkm}) - \lambda_a F^j_t F^k_s F^m_n P_{rjkm} = [\nabla_h (F^j_t F^k_s F^m_n P_{ijkm}) - \lambda_h F^j_t F^k_s F^m_n P_{ijkm}] F^i_a F^h_r. \quad (2.15)$$

PROOF: The equation (2.5) is equivalent to

$$(\nabla_a F^i_h) F^j_t F^k_s P_{ijkm} + F^i_h \nabla_a (F^j_t F^k_s P_{ijkm}) - \lambda_a F^i_h F^j_t F^k_s P_{ijkm} = 0 \quad (2.16)$$

Interchanging the indices a and h in the above equation and adding the result, thus obtained in (2.16), we get after using (1.8),

$$F^i_h [\nabla_a (F^j_t F^k_s P_{ijkm}) - \lambda_a F^j_t F^k_s P_{ijkm}] + F_a^i [\nabla_h (F^j_t F^k_s P_{ijkm}) - \lambda_h F^j_t F^k_s P_{ijkm}] = 0 \quad (2.17)$$

Transvecting the above equation by F^h_r and using (1.1), we get the condition (2.14). The proof of the condition (2.15) is similar to the proof of the condition (2.14).

3. Almost kaehlerian projective symmetric space

DEFINITION(3.1): An almost Kaehler space, for which the holomorphically projective curvature tensor P^h_{ijk}, satisfies

$$\nabla_a P^h_{ijk} = 0, \text{ or equivalently } \nabla_a P_{ijkm} = 0, \quad (3.1)$$

will be called an almost Kaehlerian projective symmetric space in the sense of Cartan.

DEFINITION(3.2): An almost Kaehler space, for which the holomorphically projective curvature tensor P^h_{ijk}, satisfies

$$\nabla_a (F^i_h P_{ijkm}) = 0, \quad (3.2)$$

will be called an almost Kaehlerian projective symmetric space of the first order and first kind.

DEFINITION(3.3): An almost Kaehler space, for which the holomorphically projective curvature tensor P^h_{ijk}, satisfies

$$\nabla_a (F^i_h F^j_t P_{ijkm}) = 0, \quad (3.3)$$

will be called an almost Kaehlerian projective symmetric space of the second order and first kind.

DEFINITION(3.4): An almost Kaehler space, for which the holomorphically projective curvature tensor P^h_{ijk}, satisfies

$$\nabla_a (F^i_h F^k_s P_{ijkm}) = 0, \quad (3.4)$$

will be called an almost Kaehlerian projective symmetric space of the second order and second kind.

DEFINITION(3.5): An almost Kaehler space, for which the holomorphically projective curvature tensor P^h_{ijk}, satisfies

$$\nabla_a (F^i_h F^j_t F^k_s P_{ijkm}) = 0, \quad (3.5)$$

and

$$\nabla_a (F^i_h F^j_t F^k_s F^m_n P_{ijkm}) = 0, \quad (3.6)$$

Will be called respectively an almost Kaehlerian projective symmetric space of the third order and the fourth order.

We, now, have the following:

THEOREM (3.1): The condition that an almost Kaehlerian space be a projective symmetric space of the first order and first kind is

$$\nabla_a P_{rjkm} - F^i_a F^h_r \nabla_h P_{ijkm} = 0. \quad (3.7)$$

PROOF: Equation (3.2) is equivalent to

$$(\nabla_a F^i_h) P_{ijkm} + F^i_h \nabla_a P_{ijkm} = 0. \quad (3.8)$$

Interchanging the indices a and h in the above equation and adding the result thus obtained in equation (3.8), we get after using (1.8),

$$F^i_h \nabla_a P_{ijkm} + F^i_a \nabla_h P_{ijkm} = 0, \quad (3.9)$$

Transvecting the above equation with F^h_r and using (1.1), we have the required condition (3.7).

THEOREM (3.2): The condition that an almost Kaehlerian space be a projective symmetric space of the second order and first kind is

$$\nabla_a (F^j_t P_{rjkm}) - F^i_a F^h_r \nabla_h (F^j_t P_{ijkm}) = 0 \quad (3.10)$$

PROOF: Equation (3.3) is equivalent to

$$(\nabla_a F^i_h) F^j_t P_{ijkm} + F^i_h \nabla_a (F^j_t P_{ijkm}) = 0 \quad (3.11)$$

Interchanging the indices a and h in the above equation and adding the result thus obtained in (3.11), we get after using (1.8),

$$F^i_h \nabla_a (F^j_t P_{ijkm}) + F^i_a \nabla_h (F^j_t P_{ijkm}) = 0 \quad (3.12)$$

Transvecting the above equation with F^h_r and using (1.1), we get the required condition (3.10).

Similarly, we can prove the following:

THEOREM (3.3): The condition that an almost Kaehlerian space be a projective symmetric space of the second order and second kind is

$$\nabla_a (F^k_s P_{rjkm}) - F^i_a F^h_r \nabla_h (F^k_s P_{ijkm}) = 0. \quad (3.13)$$

THEOREM (3.4): The conditions that an almost Kaehlerian space be a projective symmetric space of the third and fourth order are:

$$\nabla_a (F^j_t F^k_s P_{rjkm}) - F^i_a F^h_r \nabla_h (F^j_t F^k_s P_{ijkm}) = 0. \quad (3.14)$$

And $\nabla_a (F^j_t F^k_s F^m_n P_{rjkm}) - F^i_a F^h_r \nabla_h (F^j_t F^k_s F^m_n P_{ijkm}) = 0$ (3.15)

respectively.

PROOF: The equation (3.5) is equivalent to

$$(\nabla_a F^i_h) F^j_t F^k_s P_{ijkm} + F^i_h \nabla_a (F^j_t F^k_s P_{ijkm}) = 0. \quad (3.16)$$

Interchanging the indices a and h in the above equation and adding the result thus obtained in (3.16), we get after using (1.8),

$$F^i_h \nabla_a (F^j_t F^k_s P_{ijkm}) + F^i_a \nabla_h (F^j_t F^k_s P_{ijkm}) = 0. \quad (3.17)$$

Transvecting the above equation with F^h_r and using (1.1), we get the required condition (3.14).

Similarly, we can prove the condition (3.15).

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