Fuzzy economic production lot-size model under imperfect production process with cloudy fuzzy demand rate

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Abstract
The aim of the article is to develop classical economic production lot-size (EPL) model of an item produced in imperfect production process with fixed set up cost and without shortages in fuzzy environment where demand rate of an item is cloudy fuzzy number and production rate is demand dependent. In general, fuzziness of any parameter remains fixed over time but in practice, fuzziness of parameter begins to reduce as time progress because of gathering experience and knowledge. The model is solved in crisp, general fuzzy and cloudy fuzzy environment using Yager’s index method and De and Beg’s ranking index method and comparison are made for all cases. Here, the average cost function is minimized using dominance based Particle Swarm Optimization (DBPSO) algorithm to find decision for the decision maker (DM). The model is illustrated with some numerical examples and some sensitivity analyses have been done to justify the notion.

Keywords: EPL, Reliability, De and Beg's ranking index method, cloudy fuzzy number, DBPSO.

1. Introduction
In the development of economic production lot-size model, usually researchers consider the demand rate as constant in nature. In the real world, it is observed that these quantities will have little changes from the exact values. Thus in practical situations, demand variable should be treated as fuzzy in nature. Recently fuzzy concept is introduced in the production/ inventory problems. At first, Zadeh (1965) introduced the fuzzy set theory. After that, it has been applied by Bellman and Zadeh (1970) in decision making problems. Numerous researches have been done in this area. Researchers like Kaufmann and Gupta(1992), Mandal and Maiti (2002), Maiti et.al (2014), Maiti and Maiti(2006,2007), Bera and Maiti (2012), Mahata and Goswami(2007, 2013 ), De and Sana(2015) etc. have investigated extensively over this subject. Kau and Hsu(2002) developed a lot-size reorder point inventory model with fuzzy demands. In this study, a cloudy fuzzy inventory model is developed depending upon the learning from past experience. In defuzzification methods, specially on ranking fuzzy numbers, after Yager (1981), some researchers like Ezzati et at. (2012), Deng (2014), Zhang et al. (2014) and others adopted the method for ranking of fuzzy numbers based on centre of gravity. Moreover, De and Beg (2016) and De and Mahata (2016) invented new defuzzification method for triangular dense fuzzy set and triangular cloudy fuzzy set respectively. In this model, fuzziness depends upon time. As the time progress, fuzziness become optimum at the optimum time. This idea is incorporated in cloudy fuzzy environment. Till now, none has addressed this type of realistic production inventory model with cloudy fuzzy demand rate.

In the classical economic production lot-size (EPL) model, the rate of production of single item or multiple items is assumed to be inflexible and predetermined. However, in reality, it is observed that the production is influenced by the demand. When the demand increases, consumption by the customer obviously more and to meet the additional requirement of the customer, the manufactures bound to increase their production. Converse is true for reverse situation. In this connection, several researchers developed EPL models for single/multiple items considering either uniform or variable production rate (depend on time, demand and/or on hand inventory level). Bhunia and Maiti (1997), Balkhi and Benkherouf (1998), Abad (2000), Mandal and Maiti (2000) etc. developed their inventory models considering either uniform or variable production rate. However, manufacturing flexibility has become more important factor in inventory management. Different types of flexibility in manufacturing system have been identified in the
literature among which volume flexibility is the most important one. Volume flexibility of a manufacturing system is defined as its ability to be operated profitably at different output levels. Cheng (1989) first developed the demand dependent production unit cost in EPQ model; Khouja (1995) introduced volume flexibility and reliability consideration in EPQ model. Shah and Shah (2014) developed EPQ model for time declining demand with imperfect production process under inflationary conditions and reliability. 

Items are produced using conventional production process with a certain level of reliability. Higher reliability means that the products with acceptable quality are more consistently produced by the process reducing the cost of scraps, rework of substandard products, wasted materials, labor hours etc. A considerable number of research paper have been done on imperfect production by Rosenblatt and Lee(1986), Ben-Daya and Hariga(2000), Goyal et al. (2003), Maiti et al. (2006), Sana et al. (2007), Manna et al. (2014), Pal et al. (2014), etc. Recently, Manna et al. (2016) considered multi-item EPQ model with learning effect on imperfect production over fuzzy random planning horizon. Khara et al. (2017) developed an inventory model under development dependent imperfect production and reliability dependent demand. 

Use of soft computing techniques for inventory control problems is a well established phenomenon. Several authors use Genetic Algorithm (GA) in different forms to find marketing decisions for their problems. Pal et al. (2009) uses GA to solve an EPQ model with price discounted promotional demand in an imprecise planning horizon. Bera and Maiti (2012) used GA to solve multi-item inventory model incorporating discount. Maiti et al. (2009) used GA to solve inventory model with stochastic lead time and price dependent demand incorporating advance payment. Mondal and Maiti (2002), Maiti(2006,2007), Maiti et al. (2014) many other researchers uses GA in inventory control problems. Also, Bhunia and Shaikh (2015) used PSO to solve two-warehouse inventory model for deteriorating item under permissible delay in payment. Here, dominance based particle swarm optimization has been developed to solve this fuzzy inventory model.

Here, fuzzy inventory model under imperfect production process with cloudy fuzzy demand rate is developed where production rate is demand dependent. The model is solved in crisp , general fuzzy and cloudy fuzzy environment using Yager’s index method and De and Beg’s ranking index method for defuzzification and compare the results obtained in crisp, fuzzy and cloudy fuzzy environment. In this study, objective is to minimize average total cost to obtain the optimum order quantity and the cycle time using dominance based Particle Swarm Optimization (PSO) algorithm to find decision for the decision maker (DM). The model is illustrated with some numerical examples and some sensitivity analyses have been presented.

2. Definitions and Preliminaries

2.1 Normalized General Triangular Fuzzy Number (NGTFN):

A NGTFN \( \hat{A}(a_1, a_2, a_3) \) (cf. Fig-1) has three parameters \( a_1, a_2, a_3 \) where \( a_1 < a_2 < a_3 \) and is characterized by its continuous membership function \( \mu_{\hat{A}}(x) : X \rightarrow [0,1] \), where \( X \) is the set and \( x \in X \), is defined by

\[
\mu_{\hat{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
\frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\
0, & \text{otherwise}
\end{cases}
\]

(1)

![Fig-1: Membership function of a triangular fuzzy number](image)

2.2 \( \alpha \)-Cut of a fuzzy number:

A \( \alpha \) cut of a fuzzy number \( \hat{A} \) in \( X \) is denoted by \( A_{\alpha} \) and is defined as crisp set \( A_{\alpha} = \{x: \mu_{\hat{A}}(x) \geq \alpha, x \in X\} \) where \( \alpha \in [0,1] \). Here, \( A_{\alpha} \) is a non-empty bounded closed interval contained in \( X \) and it can be denoted by \( A_{\alpha} = [A_\alpha(\alpha), A_\kappa(\alpha)] \) where \( A_\alpha(\alpha) = a_1 + (a_2 - a_1)\alpha \) is the left \( \alpha \)-cut and \( A_\kappa(\alpha) = a_3 - (a_3 - a_2)\alpha \) is called the right \( \alpha \)-cut of \( \mu_{\hat{A}}(x) \) respectively.

(2)
2.3 Yager’s Ranking Index:
If \( A_L(\alpha) \) and \( A_R(\alpha) \) be the left and right \( \alpha \) cuts of a fuzzy number \( \tilde{A} \) then the Yager’s Ranking index is computed for defuzzification as
\[
I(\tilde{A}) = \frac{1}{2} \int_0^1 (A_L(\alpha) + A_R(\alpha)) \, d\alpha = \frac{1}{4} (a_1 + 2a_2 + a_3) 
\]
(3)

Also, the degree of fuzziness \( d_f \) is defined by the formula
\[
d_f = \frac{U_b - L_b}{m}
\]
where \( U_b \) and \( L_b \) are the upper and lower bounds of the fuzzy numbers respectively and \( m \) being their respective mode.

2.4 Cloudy Normalized Triangular Fuzzy Number (CNTFN) (De and Beg (2016)):
After infinite time, the normalized triangular fuzzy number \( \tilde{A} = (a_1, a_2, a_3) \) becomes a crisp singleton then fuzzy number \( \tilde{A} = (a_1, a_2, a_3) \) is called the cloudy fuzzy number. This means that both \( a_1, a_3 \to a_2 \) as \( t \to \infty \).

So, the cloudy fuzzy number takes the form
\[
\lim_{t \to \infty} \left(1 - \frac{\rho}{1+t}\right) a_2 = a_2 \quad \text{and} \quad \lim_{t \to \infty} \left(1 + \frac{\sigma}{1+t}\right) a_2 = a_2 \quad \text{for} \ 0 < \rho, \sigma < 1
\]
(4)

It is to be noted that \( \tilde{A} \to \{a_2\} \).

Its membership function becomes a continuous function of \( x \) and \( t \), defined by
\[
\mu(x, t) = \begin{cases} 
    x - a_2(1 - \frac{\rho}{1+t}), & \text{if} \ a_2(1 - \frac{\rho}{1+t}) \leq x \leq a_2 \\
    a_2 \frac{\rho}{1+t}, & \text{if} \ a_2(1 - \frac{\rho}{1+t}) \leq x \leq a_2 \\
    a_2(1 + \frac{\sigma}{1+t}) - x, & \text{if} \ a_2 \leq x \leq a_2(1 + \frac{\sigma}{1+t}) \\
    a_2 \frac{\sigma}{1+t}, & \text{if} \ a_2 \leq x \leq a_2(1 + \frac{\sigma}{1+t}) \\
    0, & \text{otherwise}
\end{cases}
\]
(5)

The graphical representation of CNTFN is appeared in the Fig-2. Let left and right \( \alpha \)-cut of \( \mu(x, t) \) from (5) denoted as \( L(\alpha, t) \) and \( R(\alpha, t) \) respectively. According to definition of \( \alpha \)-cut defined in subsection 2.2,
\[
L(\alpha, t) = a_2(1 - \frac{\rho}{1+t} + \frac{\rho \alpha}{1+t}) \quad \text{and} \quad R(\alpha, t) = a_2(1 + \frac{\sigma}{1+t} - \frac{\sigma \alpha}{1+t})
\]
(6)

2.5 De and Beg’s Ranking Index on CNTFN:
Let left and right \( \alpha \)-cut off \( \mu(x, t) \) from (5) denoted as \( L(\alpha, t) \) and \( R(\alpha, t) \) respectively. Then the defuzzification formula under time extension of Yager’s ranking index is given by
\[
J(\tilde{A}) = \frac{1}{2T} \int_{\alpha=0}^{T} \int_{t=0}^{T} \{L(\alpha, t) + R(\alpha, t)\} \, d\alpha \, dt
\]
(7)

Note that \( \alpha \) and \( t \) independent variables. Thus using (5), (6) becomes
\[
J(\tilde{A}) = \frac{a_2}{2T} \left[ 2T + \frac{\alpha - \rho}{2} \log(1+T) \right]
\]
(8)
Obviously, \( \lim_{T \to \infty} \frac{\log(1 + T)}{T} = 0 \) (Using L’Hopital’s rule) and therefore \( J(\tilde{J}) \rightarrow a_2 \) as \( T \rightarrow \infty \). Note that \( \log(1 + T) \) is taken as cloud index (CI)

In practices, \( T \) is measured in days/months.

2.6 Arithmetic Operations on Normalized General Triangular Fuzzy Number (NGTFN):

Let \( \tilde{A} = (a_1, a_2, a_3) \) and \( \tilde{B} = (b_1, b_2, b_3) \) are two triangular fuzzy numbers, then for usual arithmetic operations \(+, - , \times , \div\) respectively namely addition, subtraction, multiplication and division between \( \tilde{A} \) and \( \tilde{B} \) are defined as follows:

\( i \) \( \tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \)

\( ii \) \( \tilde{A} - \tilde{B} = (a_1 - b_1, a_2 - b_2, a_3 - b_3) \)

\( iii \) \( \tilde{A} \times \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3) \)

\( iv \) \( \frac{\tilde{A}}{\tilde{B}} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}) \), \( b_1, b_2, b_3 \neq 0 \)

\( v \) \( k \tilde{A} = (ka_1, ka_2, ka_3) \) if \( k \geq 0 \)

and \( k \tilde{A} = (ka_1, ka_2, ka_3) \) if \( k < 0 \)

3. Dominance based Particle Swarm Optimization technique (DBPSO)

During the last decade, nature inspired intelligence becomes increasingly popular through the development and utilization of intelligent paradigms in advance information systems design. Among the most popular nature inspired approaches, when task is to optimize with in complex decisions of data or information, PSO draws significant attention. Since its introduction a very large number of applications and new ideas have been realized in the context of PSO (Najafi et al., 2009; Marinakis and Marinaki, 2010). A PSO normally starts with a set of solutions \( V_i(t+1) = wV_i(t) + \mu_1 r_1 (X_{pbesti}(t) - X_i(t)) + \mu_2 r_2 (X_{gbest}(t) - X_i(t)) \) (called swarm) of the decision making problem under consideration. Individual solutions are called particles and food is analogous to optimal solution. In simple terms, the particles are flown through a multi-dimensional search space, where the position of each particle is adjusted according to its own experience and that of its neighbors. The particle \( i \) has a position vector \( (X_i(t)), velocity vector \( (V_i(t)) \), the position at which the best fitness \( X_{pbest}(t) \) encountered by the particle so far and the best position of all particles \( X_{gbest}(t) \) in current generation \( t \) In generation \( t+1 \), the position and velocity of the particle are changed to \( X_i(t+1) \) and \( V_i(t+1) \) using following rules:

\( X_i(t + 1) = X_i(t) + V_i(t + 1) \)
The parameters $\mu_1$ and $\mu_2$ are set to constant values, which are normally taken as 2, $r_1$ and $r_2$ are two random values uniformly distributed in $[0,1]$, $w$ ($0<w<1$) is inertia weight which controls the influence of previous velocity on the new velocity. Here $(X_{\text{bcest}}(t))$ and $(X_{\text{gcest}}(t))$ are normally determined by comparison of objectives due to different solutions. So for optimization problem involving crisp objective the algorithm works well. If objective value due to solution $X_i$ dominates objective value due to solution $X_j$ we say that $X_i$ dominates $X_j$. Using this dominance property PSO can be used to optimize crisp optimization problem. This form of the algorithm is named as dominance based PSO (DBPSO) and the algorithm takes the following form. In the algorithm $V_{\text{max}}$ represent maximum velocity of a particle, $B_{il}(t)$ and $B_{iu}(t)$ represent lower and upper boundary of the $i$-th variable respectively, $\text{check\_constraint}(X_i(t))$ function check whether solution $X_i(t)$ satisfies the constraints of the problem or not. It returns 1 if the solution $X_i(t)$ satisfies the constraints of the problem otherwise it returns 0.

### 3.2 Implementation of DBPSO

(a) Representation of solutions: A n-dimensional real vector $X_i=(x_{i1}, x_{i2}, \ldots, x_{in})$, is used to represent $i$-th solution, where $x_{i1}, x_{i2}, \ldots, x_{in}$ represent $n$ decision variables of the decision making problem under consideration.

(b) Initialization: N such solutions $X_i=(x_{i1}, x_{i2}, \ldots, x_{in})$, $i=1,2,\ldots, N$, are randomly generated by random number generator within the boundaries for each variable $[B_{il}, B_{iu}]$, $j=1,2,\ldots, n$. Initialize $(P(0))$ sub function is used for this purpose.

(c) Dominance property: For crisp maximization problem, a solution $X_i$ dominates a solution $X_j$ if objective value of $X_i$ is greater than that of $X_j$.

(d) Implementation: With the above function and values the algorithm is implemented using C-programming language. Different parametric values of the algorithm used to solve the model are as follows (Engelbrech, 2005), $\mu_1 = 1.49618, \mu_2 = 1.49618, w = 0.7298$.

### 4. Notations and Assumptions

The following notations and assumptions are adopted to develop the proposed inventory model.

#### 4.1 Notations

- $k$ Production rate per cycle.
- $d$ Demand rate per cycle ($d<k$).
- $r$ Production process reliability.
- $q(t)$ Instantaneous inventory level.
- $Q$ Maximum inventory level (decision variable).
- $T$ Cycle length (decision variable).
- $t_1$ Production period (decision variable).
- $c_s$ Production cost per unit.
- $c_h$ Setup cost per cycle.
- $h$ Inventory carrying cost per unit quantity per unit time.
- $Z^*$ Average total inventory cost.
- $Q^*$ Optimum value of $Q$.
- $T^*$ Optimum value of $T$.
- $Z_{t_1}^*$ Optimum value of $Z$.
- $t_1^*$ Optimum value of $t_1$.

#### 4.2. Assumptions

(i) Replenishment occurs instantaneously on placing of order quantity so lead time is zero.
(ii) The inventory is developed for single item in an imperfect production process.
(iii) Shortages are not allowed.
(iv) The time horizon of the inventory system is infinite.
(vi) The production rate $k$ is demand dependent and is of the form $k=a + bd$ (12) where $a$ and $b$ are positive constants.
At the beginning of inventory system, ambiguity of demand rate is high because the decision maker (DM) has no any definite information how many people are accepting the product and how much will be demand rate. As the time progress, DM will begin to get more information about the expected demand over the process of inventory and learn whether it is below or over expected. It is generally observed that when new product comes into the market, people will take much more time (no matter what offers/discounts have been declared or what’s the quality of product) to adopt/accept the item. Gradually, the uncertain region (cloud) getting thinner to DM’s mind. In this respect, demand rate is assumed to be cloudy fuzzy (§ 2.4).

5. Model development and analysis

\[ \frac{dq(t)}{dt} = rk - d, \quad 0 \leq t \leq t_1 \]

\[ = -d, \quad t_1 \leq t \leq T \quad \text{where } rk - d > 0 \]  (13)

with boundary condition \( q(0) = 0, q(t_1) = Q, q(T) = 0 \) \hspace{1cm} (14)

The solution of the differential equation (13) using the boundary condition (14) is given by

\[ q(t) = \begin{cases} (rk - d)t, & 0 \leq t \leq t_1 \\ d(T - t), & t_1 \leq t \leq T \end{cases} \] \hspace{1cm} (15)

The length of each cycle is

\[ T = \frac{Q}{rk - d} + \frac{Q}{d} = \frac{Q}{rk - d} \] \hspace{1cm} (16)

Total holding cost for each cycle is given by

\[ H_1(Q, r, k) = \int_0^T h q(t) dt = \int_0^{t_1} (rk - d)t dt + \int_{t_1}^T d(T - t)dt = \frac{Q^2 r k}{2d(rk - d)} \] \hspace{1cm} (17)

Total production cost per cycle is

\[ P_c(Q, r, k) = c \int_0^T k dt = kt_1 = \frac{Q}{rk - d} \] \hspace{1cm} (18)

Total cost=Production cost + Set up cost + Holding cost

\[ = c P_c(Q, r, k) + c_3 + h H_1(Q, r, k) \]

\[ = \frac{ck Q}{rk - d} + c_3 + \frac{hQ^2 r k}{2d(rk - d)} \]

Therefore, the total average cost is

\[ Z = \left[ \frac{ck Q}{rk - d} + c_3 + \frac{hQ^2 r k}{2d(rk - d)} \right] / T \]

\[ = \frac{cd}{r} + \frac{c_3}{T} + \frac{hT(rk - d)d}{2r} \]

\[ = \frac{cd}{r} + \frac{c_3}{T} + \frac{hdT(a r + (b r - 1) d)}{2(a + b d)r} \]
Hence, our problem is given by

\[
\text{Minimize } Z = \frac{cd}{r} + \frac{c_3}{T} + \frac{hdT(a r + (b r - 1)d)}{2(a + b d)r}
\]

subject to \( d(T - t_1) = (rk - d)t_1 \) i.e. \( r k t_1 = d T, Q = d(T - t_1) \) \( (19) \)

Now, the problem is reduced to minimize the average cost \( Z \) and to find the optimum value of \( Q \) and \( T \) for which \( Z(Q, T) \) is minimum and the corresponding value of \( t_1 \). The average cost is minimized by DBPSO.

5.1 Fuzzy mathematical model

Initially, when production process starts, demand rate of an item is ambiguous. Naturally, demand rate is assumed to be general fuzzy over the cycle length. Then fuzzy demand rate \( d^F \) as follows \( d^F = (d_1, d_2, d_3) \) for NGTFN.

Therefore the problem (19) becomes fuzzy problem, is given by

\[
\text{Minimize } \bar{Z}^F = \frac{c d^F}{r} + \frac{c_3}{T} + \frac{h d T \{a r + (b r - 1)d^F\}}{2(a + b d^F)r}
\]

subject to \( r k t_1 = d^F T, \bar{Q}^F = d^F (T - t_1) \) \( (20) \)

Now, using (1), the membership function of the fuzzy objective, fuzzy order quantity and fuzzy production rate under NGTFN are given by

\[
\mu_1(Z) = \begin{cases} 
\frac{Z - Z_1}{Z_2 - Z_1}, & Z_1 \leq Z \leq Z_2 \\
\frac{Z_3 - Z}{Z_3 - Z_2}, & Z_2 \leq Z \leq Z_3 \\
0, & \text{otherwise}
\end{cases}
\]

where \( Z_1 = \frac{c d_1}{r} + \frac{c_3}{T} + \frac{h d T \{a r + (b r - 1)d_1\}}{2r(a + b d_1)} \)

\( Z_2 = \frac{c d_2}{r} + \frac{c_3}{T} + \frac{h d T \{a r + (b r - 1)d_2\}}{2r(a + b d_2)} \)

\( Z_3 = \frac{c d_3}{r} + \frac{c_3}{T} + \frac{h d T \{a r + (b r - 1)d_3\}}{2r(a + b d_3)} \)

\( (21) \)

\[
\mu_2(Q) = \begin{cases} 
\frac{Q - Q_1}{Q_2 - Q_1}, & Q_1 \leq Q \leq Q_2 \\
\frac{Q_3 - Q}{Q_3 - Q_2}, & Q_2 \leq Q \leq Q_3 \\
0, & \text{otherwise}
\end{cases}
\]

where \( Q_1 = d_1 (T - t_1) \)

\( Q_2 = d_2 (T - t_1) \)

\( Q_3 = d_3 (T - t_1) \)

\( (22) \)

\[
\mu_3(k) = \begin{cases} 
\frac{k - k_1}{k_2 - k_1}, & k_1 \leq k \leq k_2 \\
\frac{k_3 - k}{k_3 - k_2}, & k_2 \leq k \leq k_3 \\
0, & \text{otherwise}
\end{cases}
\]

where \( r k_1 t_1 = d_1 T \)

\( r k_2 t_1 = d_2 T \)

\( r k_3 t_1 = d_3 T \)

\( (23) \)

The index value of the fuzzy objective, fuzzy order quantity and fuzzy production rate are respectively obtained using (2) and (3) as
\[
I(\mathcal{Z}) = \frac{1}{4} (Z_i + 2Z_j + Z_k)
\]
\[
= \frac{c(d_i + 2d_j + d_k)}{4r} + \frac{c_3}{T} + \frac{hT}{8r} \left[ d_i \left\{ a r + (b r - 1)d_i \right\} + \frac{2d_j \left\{ a r + (b r - 1)d_j \right\}}{a + b d_j} + \frac{d_k \left\{ a r + (b r - 1)d_k \right\}}{a + b d_k} \right]
\]
\[
I(\mathcal{Q}) = \frac{1}{4} (Q_i + 2Q_j + Q_k) = \frac{(T - t_i)}{4} (d_i + 2d_j + d_k)
\]
\[
I(\mathcal{K}) = \frac{1}{4} (k_i + 2k_j + k_k) = \frac{T}{4r t_i} (d_i + 2d_j + d_k) \{ u \sin g (21), (22) and (23) \} \quad (24)
\]

5.1.1 Particular cases

Subcase-4.1.1.1: If \( d_i, d_j, d_k \rightarrow d \) then \( I(\mathcal{Z}) \rightarrow \frac{cd}{r} + \frac{c_3}{T} + \frac{hT (a r + (b r - 1)d)}{2(a + b d)r} \)

\[
I(\mathcal{Q}) \rightarrow d(T - t_i)
\]

and \( I(\mathcal{K}) \rightarrow \frac{dT}{r t_i} \)

This is a classical EPQ model with process reliability \( r \).

Subcase-4.1.1.2 If \( r \rightarrow 1, b \rightarrow 0 \) then \( I(\mathcal{Z}) \rightarrow \frac{cd}{r} + \frac{c_3}{T} + \frac{hT (a - d)}{2a} \)

\[ I(\mathcal{Q}) \rightarrow d(T - t_i) \]

\[ I(\mathcal{K}) \rightarrow \frac{dT}{t_i} \]

Also, this is classical EPQ model with production rate \( a \).

5.2 Cloudy fuzzy mathematical model

Initially, when production process starts, demand rate of an item is ambiguous. As the time progresses, hesitancy of demand rate tends to certain demand rate over the cycle length. Then fuzzy demand rate \( \mathcal{Q} \) becomes cloudy fuzzy following the equation (4)

Now, using (5), the membership function of the fuzzy objective, fuzzy order quantity and fuzzy production rate under CNTFN are given by

\[
\bigg\{ \begin{array}{l}
Z_{11} = \frac{c d}{r} + \frac{c_3}{T} + \frac{hT (a + (b - 1)d)}{2r(a + b d)} \\
Z_{12} = \frac{c d}{r} + \frac{hT (a + (b - 1)d)}{2r(a + b d)} \\
Z_{13} = \frac{c d}{r} + \frac{hT (a + (b - 1)d)}{2r(a + b d)} \\
\end{array} \bigg\}
\]

(25)
\[ \chi_2(Q,T) = \begin{cases} \frac{Q - Q_{11}}{Q_{12} - Q_{11}}, & Q_{11} \leq Q \leq Q_{12} \\ \frac{Q_{13} - Q}{Q_{13} - Q_{12}}, & Q_{12} \leq Q \leq Q_{13} \\ 0, & \text{otherwise} \end{cases} \]

where

\[ Q_{11} = d(1 - \frac{\rho}{1+T})(T - t_1) \]
\[ Q_{12} = d(T - t_1) \]
\[ Q_{13} = d(1 + \frac{\sigma}{1+T})(T - t_1) \]

(26)

\[ \chi_3(k,T) = \begin{cases} \frac{k - k_{11}}{k_{12} - k_{11}}, & k_{11} \leq k \leq k_{12} \\ \frac{k_{13} - k}{k_{13} - k_{12}}, & k_{12} \leq k \leq k_{13} \\ 0, & \text{otherwise} \end{cases} \]

where

\[ k_{11} = d(1 - \frac{\rho}{1+T}) \frac{T}{r t_1} \]
\[ k_{12} = \frac{dT}{r t_1} \]
\[ k_{13} = d(1 + \frac{\sigma}{1+T}) \frac{T}{r t_1} \]

(27)

Using (7) the index value of the fuzzy objective, fuzzy order quantity and fuzzy production rate are respectively are given by

\[ J(\tilde{Z}) = \frac{1}{4\tau} \int_{T=0}^{T} (Z_{11} + 2Z_{12} + Z_{13}) dT \]

\[ = \frac{1}{4\tau} \int_{0}^{\tau} \left[ \frac{cd}{r} \left(4 + \frac{\sigma - \rho}{1+T} + \frac{4c_3}{T}\right)dT \right] + \frac{1}{4\tau} \int_{0}^{\tau} \frac{hdT}{2r} \left[(1 - \frac{\rho}{1+T}) \frac{ar + (br - 1)d(1 - \frac{\rho}{1+T})}{a + bd(1 + \frac{\sigma}{1+T})} + 2 \frac{ar + (br - 1)d}{a + bd^2} + (1 + \frac{\sigma}{1+T}) \frac{ar + (br - 1)d(1 - \frac{\rho}{1+T})}{a + bd^2} \right] dT \]

(28)

\[ = I_1 + \frac{hd}{8\tau r} (I_2 + I_3 + I_4) \]

The expression of \( I_1, I_2, I_3 \) and \( I_4 \) are given in Appendix-1

\[ J(\tilde{Q}) = \frac{1}{\tau} \int_{0}^{\tau} \left[ \frac{1}{4} (Q_{11} + 2Q_{12} + Q_{13}) dT \right] = \frac{d}{4\tau} \int_{0}^{\tau} \left[ 4 + \frac{\sigma - \rho}{1+T} \right] (T - t_1) dT \] [Using (26)]

(29)

\[ J(\tilde{k}) = \frac{1}{\tau} \int_{0}^{\tau} \left[ \frac{1}{4} (k_{11} + 2k_{12} + k_{13}) dT \right] = \frac{1}{4\tau} \int_{0}^{\tau} \frac{d}{rt_1} \left[ 4 + \frac{(\sigma - \rho)}{1+T} \right] dT \] [Using (27)]

(30)

5.2.1 Stability analysis and particular cases

(i) If \( \rho, \sigma \to 0 \) then \( p \to q \) and \( u \to v \) Also, \( I_2 \to \frac{p}{2u} \tau^2 \), \( I_4 \to \frac{p}{2u} \tau^2 \), \( I_3 = \frac{p}{u} \tau^2 \)
So, \( J(\tilde{Z}) \rightarrow \frac{cd}{r} + 3\ln \frac{r}{\varepsilon} + \frac{hd}{4r\tau} \frac{pr^2}{u} \), \( J(\tilde{F}) \rightarrow d(\frac{\tau}{2} - t_f) \), \( J(\tilde{K}) \rightarrow \frac{d}{2rt_f} \).

(ii) If \( \rho, \sigma \rightarrow 0 \) then the model reduces to (i). The above expressions deduced in (i) are in the form of classical EPQ model. Thus we choose \( \varepsilon \) in such a way that above expressions reduced to classical EPQ model.

Hence, \( \frac{1}{T} = \frac{1}{\tau} \ln \frac{r}{\varepsilon} \Rightarrow T = \frac{\tau}{2} \).

From these, we get \( \varepsilon = \frac{2T}{e^2} \). Also, if \( \tau = 2 \) then \( T = 1 \). Hence, \( \varepsilon \rightarrow 2e^{-2} << 1 \)

Since \( 2 < e \Rightarrow \frac{2T}{e^2} < \frac{T}{2} \Rightarrow e < \frac{T}{2} \).

6. Numerical Illustration

The following values of inventory parameters are used to calculate the minimum values of average cost function \( (Z') \) along with the optimum inventory level \( (Q') \) and optimum cycle length \( (T') \):

\( a = 100 \), \( b = 1.22 \), \( c = 300 \) , \( h = 1.5 \) per unit, \( c = 3 \) per unit, \( r = 8 \), \( d = 500 \) units for the crisp model; for fuzzy model demand rate \( <d_1, d_2, d_3> = <460, 500, 600> \) units keeping other inventory parameters are same as taken in crisp model and that for the cloudy fuzzy model, \( \sigma = 0.16 \), \( \rho = 0.13 \), \( \varepsilon = 0.6 \). Optimum results are obtained via dominance based particle swarm optimization and presented in Table-1.

<table>
<thead>
<tr>
<th>Model</th>
<th>( t_1 ) (months)</th>
<th>( t_2 ) (months)</th>
<th>( Q') units</th>
<th>( Z'(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crisp</td>
<td>1.5</td>
<td>1.704</td>
<td>102.00</td>
<td>2125.56</td>
</tr>
<tr>
<td>Fuzzy</td>
<td>1.9</td>
<td>2.58</td>
<td>134.30</td>
<td>2164.60</td>
</tr>
<tr>
<td>Cloudy</td>
<td>1.85</td>
<td>2.22</td>
<td>183.03</td>
<td>2115.33</td>
</tr>
<tr>
<td>Fuzzy</td>
<td>2.22</td>
<td>183.03</td>
<td>2115.33</td>
<td>2115.33</td>
</tr>
</tbody>
</table>

From the above results, it has been observed that minimum cost is obtained in cloudy fuzzy model and the value of optimum cost \( Rs. 2115.33 \) after the completion 2.22 months. In cloudy fuzzy environment degree of fuzziness is less than the general triangular number as the hesitancy of fuzzy gradually decreases due to the taking experience over time.
6.1 Sensitivity Analysis of Cloudy Fuzzy Model

Table-2: Sensitivity analysis for cloudy fuzzy model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>% change</th>
<th>Average cost ( z' )</th>
<th>( \frac{(z - z') \times 100}{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>-15%</td>
<td>1833.44</td>
<td>-13.32</td>
</tr>
<tr>
<td></td>
<td>-10%</td>
<td>1927.49</td>
<td>-8.88</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
<td>2021.45</td>
<td>-4.44</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>2209.13</td>
<td>4.43</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>2302.87</td>
<td>8.86</td>
</tr>
<tr>
<td></td>
<td>15%</td>
<td>2396.55</td>
<td>13.29</td>
</tr>
<tr>
<td>a</td>
<td>-15%</td>
<td>2099.51</td>
<td>-0.75</td>
</tr>
<tr>
<td></td>
<td>-10%</td>
<td>2104.86</td>
<td>-0.49</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
<td>2110.13</td>
<td>-0.25</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>2120.45</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>2125.51</td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td>15%</td>
<td>2130.48</td>
<td>0.69</td>
</tr>
<tr>
<td>b</td>
<td>-15%</td>
<td>2006.4</td>
<td>-5.15</td>
</tr>
<tr>
<td></td>
<td>-10%</td>
<td>2046.12</td>
<td>-3.27</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
<td>2082.28</td>
<td>-1.56</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>2145.66</td>
<td>1.43</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>2173.58</td>
<td>2.75</td>
</tr>
<tr>
<td></td>
<td>15%</td>
<td>2199.39</td>
<td>3.97</td>
</tr>
<tr>
<td>c</td>
<td>-15%</td>
<td>2108.56</td>
<td>-0.32</td>
</tr>
<tr>
<td></td>
<td>-10%</td>
<td>2110.82</td>
<td>-0.21</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
<td>2113.07</td>
<td>-0.11</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>2122.09</td>
<td>0.32</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>2128.87</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td>15%</td>
<td>2135.63</td>
<td>0.96</td>
</tr>
<tr>
<td>cₐ</td>
<td>-15%</td>
<td>1833.27</td>
<td>-13.37</td>
</tr>
<tr>
<td></td>
<td>-10%</td>
<td>1927.29</td>
<td>-8.9</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
<td>2021.31</td>
<td>-4.44</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>2209.35</td>
<td>4.44</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>2303.37</td>
<td>8.89</td>
</tr>
<tr>
<td></td>
<td>15%</td>
<td>2397.38</td>
<td>13.33</td>
</tr>
<tr>
<td>h</td>
<td>-15%</td>
<td>2100.38</td>
<td>-0.71</td>
</tr>
<tr>
<td></td>
<td>-10%</td>
<td>2105.36</td>
<td>-0.47</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
<td>2110.35</td>
<td>-0.23</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>2120.31</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>2125.28</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>15%</td>
<td>2130.28</td>
<td>0.71</td>
</tr>
</tbody>
</table>

Using the above numerical illustration, the effect of under or over estimation of various parameters on average cost is studied. Here using
\[
\Delta z = \frac{(z' - z) \times 100}{z}
\]
where \( z \) is the true value and \( z' \) is the estimated value. The sensitivity analysis is shown by increasing or decreasing the parameters by 5%, 10% and 15%, taking one at a time and keeping the others as true values. The results are presented in Table-2.

It is seen from the Table-3 that the parameters \( d \) and \( c \) are highly sensitive. For the changes of demand at -15%, average inventory cost reduces to -13.32% and for 15%, the average inventory cost increases at +13.29% respectively. Also the same results observed for the changes of unit production cost. These phenomena agree with reality. But for the changes of \( a, b, c, h \) from -15% to +15%, there are moderately variations on the average cost. This sensitivity table reveals that the observations done on inventory model are more realistic and more practicable.

6.2 Effect of changing cycle time

Comparing the results obtained in crisp, general fuzzy and cloudy fuzzy environment, it has been observed from the graphical illustration (Fig-3) that cloudy fuzzy model predicts the minimum cost 2068.57 ($) and the minimum cost is obtained at cycle time 4 months which is shown in Fig-4. In Fig-4, the curve shown U shape pattern under the cloudy fuzzy model. So the curve is convex. So, it is interesting to note that cloudy fuzzy model is more reliable.
6.3 Effect of changing reliability

Reliability is the most important factor in manufacturing system as reliability defined to be capability of manufacturing units without breakdown of the system. It has been observed from the graphical illustration (Fig-5) that as the reliability increases, average cost gradually decreases because increase of reliability resulted in increase of production rate. So, cost of finished good consistently decreases. Also, the performance level as measured by reliability can significantly improved the manufacturing system. Since the present is minimization problem, so average cost decreases with the increase of reliability.
6.4 Comparison of average cost under different cycle time
It has been observed that difference of average inventory cost of crisp model as well as general fuzzy model with respect to cloudy fuzzy model for different value of cycle time are shown in Table-3.

From this Table-3, it is seen that cloudy fuzzy model giving the minimum average inventory cost at time 4 months which is the better choice of inventory practitioner as well as decision maker.

Table-3: Average cost under different model

<table>
<thead>
<tr>
<th>Cycle time T</th>
<th>Crisp model</th>
<th>General fuzzy model</th>
<th>Cloudy fuzzy model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_1^*$</td>
<td>$Q^*$</td>
<td>$Z^*$</td>
</tr>
<tr>
<td></td>
<td>$t_1^*$</td>
<td>$Q^*$</td>
<td>$Z^*$</td>
</tr>
<tr>
<td>3</td>
<td>2.64</td>
<td>179.58</td>
<td>2109.86</td>
</tr>
<tr>
<td>4</td>
<td>3.52</td>
<td>239.46</td>
<td>2129.58</td>
</tr>
<tr>
<td>5</td>
<td>4.41</td>
<td>299.29</td>
<td>2159.47</td>
</tr>
<tr>
<td>6</td>
<td>5.20</td>
<td>359.15</td>
<td>2194.36</td>
</tr>
<tr>
<td>7</td>
<td>6.11</td>
<td>419.01</td>
<td>2232.11</td>
</tr>
<tr>
<td>8</td>
<td>7.04</td>
<td>478.87</td>
<td>2271.65</td>
</tr>
<tr>
<td>9</td>
<td>7.92</td>
<td>538.73</td>
<td>2312.33</td>
</tr>
<tr>
<td>10</td>
<td>8.81</td>
<td>598.59</td>
<td>2353.95</td>
</tr>
</tbody>
</table>

7. Conclusion and future research
In this paper, fuzzy inventory model under imperfect production process with cloudy fuzzy demand rate is developed where production rate is demand dependent. The model is solved in crisp, general fuzzy and cloudy fuzzy environment using Yager’s index method and De and Beg’s ranking index method using new defuzzification method and the results obtained in crisp, fuzzy and cloudy fuzzy environment are compared. For the first time, this type of inventory model has been successfully solved by DBPSO in cloudy fuzzy environment. Further extension of this model can be done considering some realistic situation such as multi-
item, quantity discount, price and reliability dependent, learning effect etc. Moreover, in future, this model can be formulated with random planning horizon, fuzzy planning horizon in stochastic, fuzzy stochastic environments.

Acknowledgements
The author would like to thank the University Grant Commission (UGC), India for financial support under the research grant PSW-132/14-15(ERO).

Reference


Appendix-1: The expression of $I_1, I_2, I_3$ and $I_4$ are given below.

\[
I_1 = \frac{1}{4\tau} \int_0^\tau \left\{ \frac{c}{r} \left(4 + \frac{\sigma - \rho}{1 + T} + \frac{4c_s}{T}\right) \right\} dT = \frac{c}{r} \left(1 + \frac{\sigma - \rho}{4\tau} \ln(1 + \tau) + \frac{c_s}{\tau} \ln \frac{\tau}{\epsilon}\right)
\]

\[
I_2 = \int_0^\tau T \left(1 - \frac{\rho}{1 + T}\right) \left\{ \frac{ar + (br - 1)d(1 - \frac{\rho}{1+T})}{a + bd(1 + \frac{\sigma}{1 + T})}\right\} dT
\]

\[
= \int_0^\tau T \left(1 - \frac{\rho}{1 + T}\right) \frac{Ta r + (br - 1)d(1 - \rho)}{a + bd(1 + \sigma)} + a + bd(1 + \sigma) dT
\]

\[
= \int_0^\tau T \left(1 - \frac{\rho}{1 + T}\right) \frac{pT + q}{Tu + v} dT
\]

Possibility constraints, *Fuzzy sets and fuzzy systems*, 157, 52-73.


\[
\begin{align*}
\text{where } I_{21} &= p \left[ \int_0^r \frac{T^2}{u} dT + q \left( \int_0^r \frac{T}{u} dT - \rho \int_0^r \frac{T^2}{(u+v)(1+T)} dT - \rho q \int_0^r \frac{T}{(u+v)(1+T)} dT \right) \right] \\
I_{22} &= q \left[ \int_0^r \frac{T}{u} dT = \frac{\tau - \ln\left(\frac{v + \tau u}{v}\right)}{u} \right] \\
I_{23} &= \rho p \left[ \int_0^r \frac{T^2}{(u+v)(1+T)} dT = \frac{\tau - \ln(1+\tau)}{u} \right] \\
I_{24} &= \rho q \left[ \int_0^r \frac{T}{(u+v)(1+T)} dT = \frac{\tau - \ln|1+\tau|}{u} \right] \\
I_3 &= \frac{2}{a + b d} \left( \int_0^r \frac{T}{u} dT \right) \\
I_4 &= \left\{ \frac{a r + (b r - 1) d (1 + \sigma)}{1 + T} \right\} \left\{ a + b d (1 - \frac{\rho}{1+T}) \right\} \\
&= \left[ \frac{T}{1+1+T} \right] \left( \frac{T(a r + (b r - 1) d) + a r + (b r - 1) d (1+\sigma)}{T(a + b d) + a + b d (1 - \rho)} \right) \\
&= \left[ \frac{p T + y}{u+T} \right] \\
\end{align*}
\]

\[
\begin{align*}
\text{where } I_{41} &= \left[ \frac{T^2}{u} \right] \\
I_{42} &= \left[ \frac{T}{u} \right] \\
I_{43} &= \left[ \frac{T^2}{s} \right] \\
I_{44} &= \left[ \frac{T}{s} \right] \\
\end{align*}
\]