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More on Multifunctions

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Abstract.

In these paper semi-pre continuous multifunctions has been studied in the functional point of view. Also the results showing how a semipro-continuous multifunction transfer topologies from domain set to range set, have been obtained.

Key words: s.p.o., spcl, lspc, uspc.

1. Introduction.

The term multifunction is an abbreviation of the word multi-valued function.R.E. Smithson [8] has elegantly pointed out the difference between the theory of single valued function that of multifunction. Upper semi-continuous multifunctions and lower semi-continuous multifunctions are introduced by Kuratowski [5] and Bouligand [4] independently. A large number of topologists have extended and generalized many results of single valued function to multifunction. Andrejevic [1] introduced the notion of semi-preopen sets. Przemski [7] defined semi-continuity as a single valued multifunction. Bandyopadhyay [2] introduced semi-precontinuous multifunction utilising semi-preopen set. Further properties of this space including graph condition has been obtained in this paper.

2. Materials and Methods.

Throughout the paper (X, τ) or simply X always denotes nontrivial topological spaces. The closure (resp. interior) of the subset A is denoted by Cl (A) (resp. Int (A)).

Definition 2.1. [1] In (X, τ), $A \subset X$ is called

(i) A semi-preopen set (briefly s.p.o. set) iff $A \subset Cl (Int (Cl (A)))$.

The family of all s.p.o. sets is denoted by SPO(X). For each $x \in X$, the family of all s.p.o. sets containing x is denoted by SPO(X, x).

Definition 2.2.[1] The complement of a s.p.o. set is called semi-preclosed. Equivalently a set F is semi-preclosed iff $Int (Cl (Int (A))) \subset F$. The family of all semi-preclosed sets is denoted by SPF (X).

Definition 2. 3.[6] In (X, τ), $A \subset X$ is called an α-set iff $A \subset Int (Cl (Int (A)))$.

Definition 2.4. [7] A single valued function f: X $\to$Y is said to be semi- precontinuous if the inverse image of every open set in Y is semi-preopen in X.

Definition 2.5 .[1] The semi-preclosure of $A \subset X$ is denoted by spcl (A) and is defined by spcl (A) = $\cap$ {B : B is semi-preclosed and $B \supset A$}.

For a multifunction $F : X \to Y$, $F^+ [B]$ and $F^- [B]$ respectively denote the upper and lower inverses of the set $B \subset Y$, where $F^+ [B] = \{x \in X : F (x) \subset B\}$ and $F^- [B] = \{x \in X : F (x) \cap B \neq \phi\}$.In this paper, multifunctions are denoted by upper case letters and single valued functions are denoted by lower case letters.

The following lemma have been frequently utilised to obtain desired results.

Lemma 2.1. [3] For any multifunction $F: X \to Y$

1. $F^+ [Y - B] = X - F^- [B], B \subset Y$;
2. $F^- [Y - B] = X - F^+ [B], B \subset Y$;
3. $A \subset F^+ [F [A]], A \subset X$;
4. $F [F^+ [B]] \subset B, B \subset Y$;
5. $B \subset C \subset Y \Rightarrow F^+ [B] \subset F^+ [C] , F^- [B] \subset F^- [C]$;
6. $A \subset C \subset X \Rightarrow F [A] \subset F [C]$;

7. $F[A] \cap B \subset F[A \cap F^{-}[B]]$, $A \subset X$, $B \subset Y$;
8. If $A_\alpha \subset X\ \forall\ \alpha \in \Lambda$, then $F[\cup_{\alpha \in \Lambda} A_\alpha] = \cup_{\alpha \in \Lambda} F[A_\alpha]$.

Definition 2.6. [3] A multifunction $F: X \to Y$ is sad to be

1. upper semi-continuous (briefly usc) iff for each closed set $A \subset Y$, $F^{-}[A]$ is a closed subset of X,
2. lower semi-continuous (briefly lsc) iff for each open set $U \subset Y$, $F^{-}[U]$ is an open subset of X,
3. continuous iff F is both lsc and usc

Definition 2.7[2]. A multifunction $F : X \to Y$ is termed upper semi-precontinuous (resp. lower semi-precontinuous), briefly uspc (resp. lspc), iff for each closed (resp. open) set

$A \subset Y$, $F^{-}[A] \in SPF(X)$ (resp. $F^{-}[A] \in SPO(X)$).

Definition 2.8[2]. A multifunction $F : X \to Y$ is semi-precontinuous (briefly spc) iff F is both uspc and lspc.

Definition 2.9[2]. A multifunction $F : X \to Y$ is said to be

1. uspc at $x_0 \in X$ iff for each $V \in \Sigma(F(x_0))$ in Y there is a $U \in SPO(X, x_0)$ with $F(x) \subset V \quad \forall\ x \in U$;
2. uspc iff F is uspc at each point of X;
3. lspc at $x_0 \in X$ iff every open set $V \subset Y$ with $F(x_0) \cap V \neq \phi$, there exists a $U \in SPO(X, x_0)$ such that $F(x) \cap V \neq \phi\ \forall\ x \in U$;
4. lspc iff F is lspc at each point of X;
5. vxspc iff F is both lspc and uspc.

Defintion2.10 [8] For a multifunction graph map G_F of F is the multifunction $G_F : X \to X \times Y$ defined as $G_F(x) = \{x\} \times F(x)$.

3.Results and Discussions.

Theorem 3.1. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is uspc iff G_F is uspc.

Proof. Let F be uspc at $x \in X$ and $W \in \Sigma(G_F(x))$.Then $W \in \tau \times \sigma$ and $G_F(x) \subset W$.$\Rightarrow \{x\} \times F(x) \subset W$. So there exist $U \in \tau$, $V \in \sigma$ such that $U \times V \subset W$and $\{x\} \times F(x) \subset U \times V$.This implies $x \in U$ and $F(x) \subset V$. Since F is uspc at x and $V \in \Sigma(F(x))$ there exists $G \in SPO(X, x)$ such that $F(y) \subset V \quad \forall\ y \in G$. Let $R = U \cap G$. Then by the fact that every open set is an α-set it follows that $R \in SPO(X, x)$. Clearly $R \subset G$. So, $F(y) \subset V \quad \forall\ y \in R \Rightarrow \{y\} \times F(y) \subset R \times V \quad \forall\ y \in R \Rightarrow G_F(y) \subset R \times V \subset U \times V \subset W \quad \forall\ y \in R$. Hence G_F is uspc at x.

Conversely G_F is uspc at $x \in X$. We show that F is uspc at x. Let $V \in \Sigma(F(x))$. Then $F(x) \subset V$. So, $\{x\} \times F(x) \subset X \times V \Rightarrow G_F(x) \subset X \times V$. …(1) Clearly $X \times V \in \tau \times \sigma$ and so, by (1), $X \times V \in \Sigma(G_F(x))$.Now the uspc of G_F at x gives a $U \in SPO(X, x)$ such that $G_F(y) \subset X \times V \quad \forall\ y \in U \Rightarrow \{y\} \times F(y) \subset X \times V \quad \forall\ y \in U \Rightarrow F(y) \subset V \quad \forall\ y \in U$ which assures that F is uspc at x.

Theorem 3.2. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lspc iff G_F is lspc.

Proof. LetF be lspc at x. Let $W \in \tau \times \sigma$ be such that $G_F(x) \cap W \neq \phi$ So, there exist $U \in \tau$, $V \in \sigma$ such that $U \times V \subset W$ and $(\{x\} \times F(x)) \cap (U \times V) = \phi \Rightarrow \{x\} \cap U \neq \phi$ and $F(x) \cap V \neq \phi$.The lspc of F at x gives the existence of $G \in SPO(X, x)$ such that $F(y) \cap V \neq \phi \quad \forall\ y \in G$.Let $R = U \cap G$. Then, $R \in SPO(X, x)$. Clearly $R \subset G$. So, $F(y) \cap V \neq \phi \quad \forall\ y \in R \Rightarrow (\{y\} \times F(y)) \cap (R \times V) \neq \phi \Rightarrow G_F(y) \cap (R \times V) \neq \phi$.Now $R \times V \subset U \times V \subset W$ and this yields that $G_F(y) \cap W \neq \phi \quad \forall\ y \in R$ whence one infers that G_F is lspc at x.

Let G_F be lspc at x and $V \in \sigma$ with $F(x) \cap V \neq \phi$ …(2) . Now $X \times V \in \tau \times \sigma$. Also, by (2), $(\{x\} \times F(x)) \cap (X \times V) \neq \phi \Rightarrow G_F(x) \cap (X \times V) \neq \phi$. The lspc of G_F at x ensures the existence of a $U \in SPO(X, x)$ such that $G_F(y) \cap (X \times V) \neq \phi\ \forall\ y \in U \Rightarrow (\{y\} \times F(y)) \cap (X \times V) \neq \phi \quad \forall\ y \in U \Rightarrow F(y) \cap V \neq \phi\ \forall\ y \in U \Rightarrow$ F is lspc at x.

Definition 3.1. A multifunction $F : X \to Y$ is said to be punctually compact (resp. connected) iff for each $x \in X$, $F(x)$ is compact (resp. connected).

Definition 3.2. Two subsets A and B of X are termed sp-separated iff $A \cap spcl(B) = \phi = spcl(A) \cap B$. X is said to be sp-connected if it is not the union of two non empty sp-separated subsets.

Theorem 3.3. Let $F : X \to Y$ be uspc and punctually compact. If $A \subset X$ is sp-compact, then $F[A]$ is compact.

Proof. Let $\tilde{U}$ be an open cover of $F[A]$ and $x \in A$. Then $F(x) \subset F[A] \subset \cup\{U : U \in U\}$

$\Rightarrow$ U is a cover of $F(x)$ by open sets in Y. Since F is punctually compact $F(x)$ is compact. So, we obtain a finite number of members $U_1(x), U_2(x), \ldots, U_m(x)$ from $\tilde{U}$ such that $F(x) \subset \cup\{U_i(x), i=1,2,\ldots,m\}$. Now $\cup\{U_i(x) : i=1,2,\ldots,m\}$ is open in Y and contains $F(x)$. The uspc of F then provides a $V_x \in SPO(X, x)$ such that $F(y) \subset \cup\{U_i(x): i=1,2,\ldots,m\}\ \forall y \in V_x \Rightarrow \cup F(y) \subset \cup\{U_i(x) : i=1,2,\ldots,m\} \Rightarrow F[V_x] \subset \cup\{U_i$

(x) i=1,2,…,m }Thus for each x ∈ A we obtain a V_x ∈ SPO (X, x) and m members U_1 (x). U_2 (x), …, U_m (x) from U with F [V_x] ⊂ ∪{U_i (x) i=1,2,…,m }.Clearly, A ⊂ ∪ V_x⇒ {V_x : x ∈ A} is a cover of A by s.p.o. sets in X. The sp-compactness of A then assures the existence of a finite number of points x_1, x_2, …, x_n in A such that A ⊂ ∪{Vx_i : i=1,2,…,n}. F [A] ⊂ F [∪{Vx_i : i=1,2,…,n}] = ∪{F [Vx_i]: i=1,2,…,n} ⊂ ∪{{∪{U_i(x_j) :i= 1,2,…,m}} :j=1,2,…,n}where U_i (x_j) ∈ U for i = 1, 2, …, m,j=1,2,…,n. This shows that {U_i (x_j) : i = 1, 2, …, m , j = 1, 2, …, n}is a finite subcover of U . Hence F [A] is compact.

Theorem 3.4. Let F : X → Y be lspc and punctually connected. If A ⊂ X is sp-connected then F [A] is connected.

Proof. Suppose the theorem is false. Then there exist two non empty sets G_1, G_2, in Y such that F [A] = G_1 ∪ G_2 with G_1 ∩ Cl (G_2) = ϕ = Cl (G_1) ∩ G_2.Let B_i = {x ∈ A : F (x) ⊂ G_i } i = 1,2.
Thus x ∈ A ⇒ F (x) ⊂ F [A] ⇒ F (x) ⊂ G_1 ∪ G_2.Since F is punctually connected, F (x) is connected. Hence from above x ∈ A ⇒ F (x) ⊂ G_1 or F (x) ⊂ G_2⇒ x ∈ B_1 or x ∈ B_2 ⇒ x ∈ B_1 ∪ B_2.Consequently, A ⊂ B_1 ∪ B_2.The reverse inclusion B_1 ∪ B_2 ⊂ A is obvious and hence
A = B_1 ∪ B_2. The non-emptiness of G_i assures that B_i (i = 1, 2) is nonempty. Let x ∈ B_1 and y ∈ F (x) ⊂ G_1.Since G_1 ∩ Cl (G_2) = ϕ, there exists V ∈ Σ (y) such thatV ∩ G_2 = ϕ.
Clearly, F (x) ∩ V ≠ ϕ. The lspc of F at x now gives the existence of a U ∈ SPO (X, x) with F (z) ∩ V ≠ ϕ ∀ z ∈ U.Let p ∈ A ∩ U. From the disjointness of V with G_2, and non-void intersection of F (p) with V we infer that F (p) ∩ G_1 ≠ ϕ.So, from the fact that F (p) ⊂ F [A] = G_1 ∪ G_2 and connectedness of F (p) we conclude thatF (p) ⊂ G_1 and F (p) ∩ G_2 = ϕ.The second relation implies that ⇒ p ∉ B_2.Since p is any member of A ∩ U, it follows that (A ∩ U) ∩ B_2 = ϕ ,whence U ∩ B_2 = A ∩ (U ∩ B_2) = ϕ.Thus for each x ∈ B_1 there exists U ∈ SPO (X, x) such that U ∩ B_2 = ϕ which induces x ∉ spcl (B_2) ∀ x ∈ B_1.

Hence B_1 ∩ spcl (B_2) = ϕ. In like manner, we obtain spcl (B_1) ∩ B_2 = ϕ.These lead to the fact that A is not sp-connected, a contradiction to the hypothesis. Hence the theorem.

4. Conclusion.

The concept of multifunctions is an active field of study but this has not found place in most text book till date...Only in 1963 C.Berge [3] discussed this function in his book. But this book suits more to the need of a person interested in Analysis than of a Topologist. Through the sustained Endeavour of a large number of renowned topologists for the last four decades the continuity conditions have been weakened yielding a variety of weak form of continuity. In this paper an attempt have been made to extend weak form of continuity conditions for single valued functions to multivalued functions.

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