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On Fixed Point Results in Generalized Metric Space

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Abstract

In this paper we prove a unique fixed point result of Kanan type map in Generalized Metric space (g.m.s.) where triangle inequality is replaced by rectangular property.

Keywords: -Generalized Metric Space , Contraction mapping, fixed point, T-Orbitally complete, Uniformly Locally Contractive mapping .

Mathematics Subject classification:- 47H10,54H25

1. Introduction

The fixed point theory has generally developed in three main directions through generalization of conditions which guarantees existence and if possible uniqueness of fixed point.

If (X,d) is a Complete metric space. $T:X\to X$ is a contraction mapping if

$$d(Tx,Ty)\le \alpha d(x,y)$$

for $0<\alpha<1$ for all $x,y\in X$ then the very important result widely known as Banach Contraction mapping principle [3] states that T has a unique fixed point. Recently in 2000 a very interesting notion of generalized metric space was developed by Branciari[2].Branciari generalized Banach Contraction Principle in Generalized metric space(g.m.s.). In 2002 P.Das. [5] generalized fixed point result in generalized metric space (g.m.s.).Later in 2009 Dorel Mihet[7] generalized Kanan fixed point theorem in this space.

2. Some preliminaries

Definition 2.1 [2] **:-** Let X be any non empty set and $d:X\times X\to R^{+}$ be a self map. Then d *is* said to be as Generalized metric or Generalized distance map if it satisfies following conditions for all $x,y,z\in X$

$(i)d(x,y)=0$ iff $x=y$

$(ii)d(x,y)=d(y,x)$ (symmetry)

$(iii)d(x,y)\le d(x,z)+d(z,w)+d(w,y)$ (quadrilateral/rectangular inequality)

A set X equipped with metric d is said to be a Generalized metric space or (g.m.s.)

Remark:- Every metric space is a generalized metric space but the converse may not always be true.

Definition 2.2 :- A metric *d* is said to have a semi triangular property if it satisfies

For all $x,y,z \in X$,

$d(x,y)<\in/2$ and $d(y,z)<\in/2$ this gives $d(x,z)<\in$ (1)

Definition 2.3 [1] :- Let X be a non empty set and f be a self map on X , then a point $x\in X$ is called a fixed point of f if $f(x)=x$.

Example 2.1 [1] If f is a self map on real numbers, defined by $f(x)=x^2-3x+4$.

Then 2 is a fixed point of f because $f(2)=2$.

Definition 2.4 [6] :- Let T be a mapping of g.m.s. (X,d) into itself is called as T-Orbitally complete iff every Cauchy Sequence

$$\{x_n\}\subseteq\{x,Tx,T^2x,T^3x,........\} \text{ for } x\in X$$

converges in X itself.

Definition 2.5:- $T:X\to X$ is called $(\in,\lambda)$ Uniformly locally contractive if it is locally contractive at all points of $x\in X$ and $\in,\lambda$ do not depend on x i.e.

$$d(x,y)<\in \Rightarrow d(Tx,Ty)<\lambda\, d(x,y) \text{ for all } x,y\in X$$

We begin by recalling the fixed point theorem of Kanan in a g.m.s. [3] stated as

Theorem 3.1 (Kanan fixed point theorem in complete metric space) Let X be a Complete metric Space and $F: X \to X$ is a mapping such that,

$$d(Fx, Fy) \le \beta\ [d(x, Fx) + d(y, Fy)], \text{for all } x, y \in X.$$

Where $\beta \in\ ]0,1[$.Then f has a Unique fixed point in X.

By taking concept of F-orbitally Complete maps then a slight generalization of 3.1 is as follows.

Theorem 3.2 [7] If (X, d) be a generalized metric space and $F: X \to X$ is a mapping such that,

$$d(Fx, Fy) \le \beta\ [d(x, Fx) + d(y, Fy)] \quad (2)$$

for all $x, y \in X$

Where $\beta \in\ ]0,1[$. If X is F-orbitally complete. Then F has a Unique fixed point in X.

In our Result we show that the existence of a fixed point for a Kanan contraction in orbitally complete Generalized metric space is actually a generalization of Theorem 3.1.

3. Main Result

Theorem 3.3 Let (X, d) be a generalized metric space , and the mapping $F: X \to X$ is uniformly locally contractive which satisfies (2).

Then F has a Unique fixed point in X.

Proof :- Let x_0 be any point in X . Let F(x_0)= x_1.

If x_1 and x_0 coincides then F(x_0)= x_1 . It shows that x_0 is a fixed Point of F. And the theorem is obviously proved.

Let us suppose that x_1 and x_0 are distinct i.e. $x_0 \ne x_1$

consider $F(x_1) = x_2, F(x_2) = x_3, \ldots\ldots\ldots,$

$F(x_n) = x_{n+1} = F^{n+1}x_0$ and $x_{n+1} \ne x_n$ for n=0,1,2,.......

In this way we have a sequence { x_n } in this manner.

Now consider

$$d(x_n, x_{n+1}) = d(Fx_{n-1}, Fx_n)$$

$$\le \beta[d(x_{n-1}, Fx_{n-1}) + d(x_n, Fx_n)]$$

(∵ by inequality (2))

$$\le \beta\ [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$\le \frac{\beta}{1-\beta}\ d(x_{n-1}, x_n)$$

Let us suppose that x_0 is not a periodic point , and if $x_n = x_0$, then We have,

$$d(x_0, Fx_0) = d(x_n, Fx_n) = d(F^n x_0, F^{n+1}x_0)$$

$$\le \frac{\beta}{1-\beta}\ d(F^{n-1}x_0, F^n x_0)$$

$$\le \left(\frac{\beta}{1-\beta}\right)^2 d(F^{n-1}x_0, F^{n-1}x_0)$$

$$\le \ldots\ldots\ldots\ldots\ldots$$

$$\le \ldots\ldots\ldots\ldots\ldots$$

$$\le \left(\frac{\beta}{1-\beta}\right)^n d(x_0, Fx_0) \quad (3)$$

Let $\alpha = \frac{\beta}{1-\beta}$ $\quad \therefore\ \alpha < 1$ and

$$(1-\alpha^n)\ d(x_0, Fx_0) \le 0$$

This gives $d(x_0, Fx_0) \le 0$

$\Rightarrow$ $Fx_0 = x_0$.It means x_0 is a fixed point of T.

Now relation (2) gives

$$d(F^n x_0, F^{n+m}x_0) \le \beta\ [d(F^{n-1}x_0, F^n x_0) + d(F^{n+m-1}x_0, F^{n+m}x_0)$$

$$\le \beta\ [\alpha^{n-1}\ d(x_0, Fx_0)\ + \alpha^{n+m-1}\ d(x_0, Fx_0)]$$

$\therefore\ d(x_n, x_{n+m}) \to 0$, as n $\to \infty$

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Hence $\{x_n\}$ is a Cauchy sequence in X . As X is a Complete $\therefore \exists$ a $p \in X$ s.t. $x_n \to p$

By using quadrilateral property of g.m.s.
We have

$$d(F_p, p) \le d(F_p, F^n x_0) + d(F^n x_0, F^{n+1} x_0) + d(F^{n+1} x_0, p)$$

$$\le \beta\, [d(p, F_p) + d(F^{n-1} x_0, F^n x_0)]$$

$$+ \alpha^n\, d(x_0, Fx_0) + d(F^{n+1} x_0, p)$$

$$\le \alpha\, d(F^{n-1} x_0, F^n x_0) + \frac{\alpha^n}{1-\beta} d(x_0, Fx_0)$$

$$+ \frac{1}{1-\beta}\, d(F^{n+1} x_0, p)$$

So taking limit as $n \to \infty$ and using the fact that ,

$d(a_{n,} y) \to d(a, y)$ and $d(x, a_n) \to d(x, a)$

whenever $\{a_{n,}\} \in X$ with $a_n \to a \in X$

This gives $p = Fp$.

i.e. p is a fixed point of F.

To prove uniqueness If possible suppose that q $\in X$ is another fixed point of F i.e $Fq = q$

Consider,

$$d(q, p) = d(Fq, Fp)$$

$$\le \beta\, [d(q, Fq) + d(p, Fp)$$

$$\le \beta\, [d(q, q) + d(p, p) = 0$$

$$\Rightarrow p = q.$$

Hence the Fixed point of F is Unique.

Theorem 3.4 [**6**] If F is an $(\in, \lambda)$ uniformaly locally contractive mapping defined on a F-orbitally complete, $\frac{\in}{2}$-chainable g.m.s. X satisfying (1). Then T has a unique fixed point in X.

Theorem 3.5 Let (X,d) be a Complete metric space and F,T: $X \to X$ be uniformly locally contractive mappings such that F is continuous , one to one.

If $\beta \in [0,1[$ and

$$d(FTx, FTy) \le \beta\, [d(Fx, FTx) + d(Fy, FTy)] \quad (x,y \in X) \tag{4}$$

Then T has a Unique Fixed point.

Proof Let x_0 be any in X . Let us define a iterative sequence $\{x_n\}$ by

$Tx_{n,} = x_{n+1}$ (likely $x_n = T^n x_0$) ,n=0,1,2......

Consider

$$d(Fx_n, Fx_{n+1}) = d(FTx_{n-1}, FTx_n)$$
$$\le \beta\, [d(Fx_{n-1}, FTx_{n-1}) + d(Fx_n, FTx_n)] \quad (\because \text{ by using (4)}) \tag{5}$$

$$\therefore\ d(Fx_n, Fx_{n+1}) \le \frac{\beta}{1-\beta} d(Fx_{n-1}, Fx_n) \tag{6}$$

$$\text{Let } \alpha = \frac{\beta}{1-\beta} \quad \therefore\ \alpha < 1$$

Using (6) and Induction, we have.

$$d(Fx_n, Fx_{n+1}) \le \alpha d(Fx_{n-1}, Fx_n) \le \alpha^2 d(Fx_{n-2}, Fx_{n-1})$$

$$\le \ldots\ldots\ldots\ldots\ldots\ldots \le \alpha^n d(Fx_0, Fx_1) \tag{7}$$

For m>n and m,n $\in$ I and by using (7) We have

$$d(Fx_m, Fx_n) \le d(Fx_m, Fx_{m-1}) + d(Fx_{m-1}, Fx_{m-2}) + \ldots\ldots\ldots + d(Fx_{n+1}, Fx_n)$$

$$\le [\alpha^{m-1} + \alpha^{m-2} + \ldots\ldots\ldots + \alpha^n]\, d(Fx_0, Fx_1)$$

$$\text{where } \alpha = \frac{\beta}{1-\beta}$$

$$= \alpha^n \frac{1}{1-\alpha} d(Fx_0, Fx_1) \tag{8}$$

Taking limit as m,n $\to \infty$ in (8), we have $\{F x_n\}$ will be a Cauchy sequence ,and since X is a Complete there exists q $\in$ X such that

$$\lim_{x \to \infty} Fx_n = q \tag{9}$$

Since F is subsequently convergent , $\therefore$ $\{x_n\}$ has a convergent subsequence.

$\therefore$ There exists p$\in$ X and $\{x_{n_k}\}_{k=1}^{\infty}$ s.t.

$\lim_{k\to\infty} x_{n_k} = p$. Given T is continuous and $\lim_{k\to\infty} x_{n_k} = p \ \therefore\ \lim_{k\to\infty} Fx_{n_k} = Fp$

$\therefore$ using (9) We have $Fp = q$

Consider $d(FTp, Fp) \le d(FTp, FT^n x_0) + d(FT^n x_0, FT^{n+1} x_0)$

$$+d(FT^{n+1}x_0, Tp)$$

$$\le \beta[d(Fp,FTp) + d(FT^n x_0, FT^n x_0)$$

$$+\alpha^n d(Fx_1, Fx_0) + d(Fx_{n+1}, Fp)$$

$$d(FTp,Fp) \le \alpha^n d(Fx_1,Fx_0) + \frac{1}{1-\beta}\left(\frac{\beta}{1-\beta}\right)^n d(Fx_1,Fx_0) + \frac{1}{1-\beta} d(Fx_{n+1},Fp) \qquad (10)$$

Taking limit as $n \to \infty$ in (10), We have $d(FTp, Fp)$ =0

Since F is one to one ∴ Tp=p. So T has a fixed point .

Uniqueness of fixed point can be easily obtained from (4).

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