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A Fixed Point Theorem for Multivalued F-Rational Contraction with δ-Distance

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Abstract

In this article, we present a fixed point theorem for multivalued rational contractions with δ -distance usingWardowski's technique on complete metricspace. Let (X, d)be a metric space and let $B(X)$be a family of all nonempty bounded subsets of X . Define $\delta: B(X) \times B(X) \rightarrow R$ by $(A, B) = \sup\{d(a, b): a \in A, b \in B$.Consideringδ-distance, it is proved that if (X, d)is a complete metric space and $T: X \rightarrow B(X)$is a multivalued certain F-rational contraction, then T has a fixed point.

Keywords:*Continuity, fixed point, multivalued mapping.*

1. Introduction

In the last thirty years, the theory of multivalued functions has advanced in a variety of ways.Fixed point theory for multivalued mappings is studied by bothLet$\mathfrak{F}$be the set of all functions$F: (0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:
Pompeiu-Hausdorff metric H [1-4], which is defined on $CB(X)$ (the family of all nonempty, closed, and bounded subsets of X), and δ-distance, which is defined on $B(X)$ (the family of all nonempty and bounded subsets of X). Using Pompeiu-Hausdorff metric, Nadler [12] introduced the concept of multivalued contraction mapping and show that such mapping has a fixed point on complete metric space. Then many authors focused on this direction [3-16]. On the other hand, Fisher [8] obtained different type of multivalued fixed point theorems defining δ-distance between two bounded subsets of a metric space X. We can find some results about this way in [2-9].In this article, we present some new multivalued fixed point results for rational contraction by considering the δ-distance. For this we use the technique, which was given by Wardowski [17]. For the sake of completeness,we will discuss its basic lines.

(F1). F is strictly increasing, i.e. for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$.
(F2)For each sequence $\{a_n\}$ of positive numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$
(F3)There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k(\alpha) = -\infty$.

Definition 1.1 (see [17]). Let (X, d) be a metric space and let $T: X \rightarrow X$be a mapping. Given $F \in \mathfrak{F}$, we say that $T$is F-contraction, if there exists $\tau > 0$such that $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \leq F\,(d\,(x, y))\,. \quad (1.1)$$

Example 1.2 (see [17]). Let $F_1: (0, \infty) \rightarrow \mathbb{R}$ be given by theformulae
$F_1(\alpha) = \ln \alpha$.
It is clear that $F_1 \in \mathfrak{F}$.Then eachself-mapping Ton a metric space (X, d) satisfying (1.1) is an F_1 -contractions such that $d(Tx, Ty) \leq e^{-\tau} d(x, y), \forall\, x, y \in X, Tx \neq Ty$.(1.2)
It is clear that for $x, y \in X$ such that $Tx = Ty$the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Therefore T satisfies Banach contraction with $L = e^{-\tau}$; thus T is a contraction.

Example 1.3 (see [17]). Let $F_2: (0, \infty) \rightarrow \mathbb{R}$ be given by the formulae
$F_2(\alpha) = \alpha + \ln \alpha$.
It is clear that $F_2 \in \mathfrak{F}$.Then each self-mapping Ton a metric space (X, d) satisfying (1.1) is an F_2 -contractions such that

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty) - d(x,y)} \leq e^{-\tau}, \forall\, x, y \in X, Tx \neq Ty. \quad (1.3)$$

We can find some different examples for the function Fbelonging to $\mathfrak{F}$ in [17]. In addition, Wardowski concluded that every F-contraction $T$is a contractive mapping, that is,

$$d(Tx, Ty) < d(x, y), \forall x, y \in X, Tx \neq \quad (1.4)$$

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Thus, every F-contraction is a continuous mapping. Also, Wardowski concluded that if $F_1, F_2 \in \mathfrak{F}$ with $F_1(\alpha) \le F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He noted that, for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$. $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing.Hence, it was obtained that every Banach contraction satisfies the contractive condition (1.3).On the other side, [17, Example 2.5] shows that the mapping T is not an F_1-contraction (Banach contraction) but still is an F_2-contraction. Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Following Wardowski, Mınak et al. [10] introduced the concept of Ciric type generalized F-contraction. Let (X, d) be a metric space and let $T: X \to X$ be a mapping. Given $F \in \mathfrak{F}$, we say that T is a Ciric type generalized F-contraction if there exists $\tau > 0$ such that $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty) \le F(m(x, y)), \quad (1.5)$$

where

$$(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right., \quad (1.6)$$

Then the following theorem was given.

Theorem 1.4 Let (X, d) be a complete metric space and let $T: X \to X$ be a Ciric type generalized F-contraction. If T or F is continuous, then T has a unique fixed point in X.

Now, we recall some definitions and notations which are used in this paper.Let (X, d) be a metric space. For $A, B \in B(X)$, we define

$$\delta(A, B) = \sup\{d(a, b): a \in A, b \in B\}$$

$$D(a, B) = \inf\{d(a, b): b \in B\} \quad (1.7)$$

If $A = \{a\}$, we write $\delta(A, B) = \delta(a, B)$ and also if $B = \{b\}$, then $\delta(a, B) = d(a, b)$. It is easy to prove that for $A, B, C \in B(X)$

$$\delta(A, B) = \delta(B, A) \ge 0,$$
$$\delta(A, B) \le \delta(A, C) + \delta(C, B),$$
$$\delta(A, A) = \sup\{d(a, b): a, b \in A\} = \text{diam} A,$$
$$\delta(A, B) = 0, \text{ implies that } A = B = \{a\} \quad (1.8)$$

Taking different functions $F \in \mathfrak{F}$ in (2.1), one gets a variety of F-contractions, some of them being already known in the literature.The following examples will certify this assertion.

If $\{A_n\}$ is a sequence in $B(X)$, we say that $\{A_n\}$ converges to $A \subseteq X$ and write $A_n \to A$ if and only if

1) $a \in A$ implies that $a_n \to a$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $\square \in \mathbb{N}$,
2) for any $\varepsilon > 0, \exists\, m \in \mathbb{N}$ such that $A_n \subseteq A_\varepsilon$ for $n > m$, where

$$A_\varepsilon = \{x \in X: d(x, a) < \varepsilon \text{ for some } a \in A\} \quad (1.9)$$

Lemma 1.6 (see [2]). Suppose $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ and (X, d) is a complete metric space. If $A_n \to A \in B(X)$ and $B_n \to B \in B(X)$, then $\delta(A_n, B_n) \to \delta(A, B)$.

Lemma 1.7 (see [2]). If $\{A_n\}$ is a sequence of nonemptybounded subsets in the complete metric space (X, d) and if $\delta(A_n, y) \to 0$ for some $y \in X$, then $\{A_n\} \to \{y\}$.

Recently,ÖzlemAcar and IshakAltun [13], introduced the following concept.

Definition 1.8 Let (X, d) be a metric space and let $T: X \to B(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction. If $F \in \mathfrak{F}$ and there exist $\tau > 0$

$$\tau + F(\delta(Tx, Ty) \le F\big(M(x_{n-1}, x_n)\big) \quad (1.10)$$

for all $x, y \in X$ with $\min\{\delta(Tx, Ty), d(x, y)\} > 0$, where

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty), D(y, Tx)]\right\} \quad (1.11)$$

And proved the following result.

Theorem 1.9 Let (X, d) be a complete matrix space and $T: X \to B(X)$ be a generalized multivalued F-contraction. If F is continuous and Tx is closed for all $x \in X$, then T has a fixed point in X .

2. Main Results

In this section, we prove a fixed point theorem for multivalued F-rational contractions with δ-distance. First, we introduce the following definition.

Definition 2.1 Let (X, d) be a metric space and let $T: X \to B(X)$ be a mapping. Then T is said to be a generalized multivalued F-rational contraction. If $F \in \mathfrak{F}$ and there exist $\tau > 0$

$$\tau + F(\delta(Tx, Ty) \le F\big(M_T(x_{n-1}, x_n)\big) \quad (2.1)$$

for all $x, y \in X$ with $\min\{\delta(Tx, Ty), d(x, y)\} > 0$, where

$$M_T(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(y, Ty)[1 + D(x, Tx)]}{1 + d(x, y)}\right\} \quad (2.2)$$

Now, our main result as follows.

Theorem 2.2 Let (X, d) be a complete matrix space and $T: X \to B(X)$ be a generalized multivalued F-rational contraction. If F is continuous and Tx is closed for all $x \in X$, then T has a fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for all $n \ge 0$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ for which $x_{n_0} = x_{n_{0+1}}$ then x_{n_0} is a fixed point of T and so the proof is completed. Thus suppose that for every $n \in \mathbb{N} \cup \{0\}$, $x_n \neq x_{n+1}$. So $d(x_n, x_{n+1}) > 0$ and $\delta(Tx_n, Tx_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then we have from (2.1) and (2.2)

$$\tau + F(d(x_n, x_{n+1})) \leq \tau + F(\delta(Tx_{n-1}, Tx_n))$$

$$\leq F(M_T(x_{n-1}, x_n)) \quad (2.3)$$

where

$M_T(x_{n-1}, x_n)$

$$= \max\left\{\begin{matrix} d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), \\ D(x_n, Tx_n), \dfrac{D(x_n, Tx_n)[1 + D((x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} d(x_{n-1}, x_n), d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}) \dfrac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} \end{matrix}\right\}$$

$$= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then from (2.3) $\tau + F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1}))$

Which is contradiction because $\tau > 0$. Therefore

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$$

and then from (2.3), we have

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) \quad (2.4)$$

So,

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau$$
$$\leq F(d(x_{n-2}, x_{n-1})) - 2\tau$$

:

$$\leq F(d(x_0, x_1)) - n\tau \quad (2.5)$$

Denote $d_n = d(x_n, x_{n+1})$ for $n = 0,1 \ldots\ldots$ Then $d_n > 0$ for all n and using (2.5), the following holds:

$$F(d_n) \leq F(d_{n-1}) - \tau$$
$$\leq F(d_{n-1}) - 2\tau$$

:

$$\leq F(d_0) - n\tau \quad (2.6)$$

From (6), we get

$$\lim_{n\to\infty} F(d_n) = -\infty$$

Thus from (F2), we have

$$\lim_{n\to\infty} d_n = 0 \quad (2.7)$$

From (F3), there exists $k \in (0, 1)$ such that:

$$\lim_{n\to\infty} d_n^k F(d_n) = 0 \quad (2.8)$$

By (2.6) the following holds for all $n \in \mathbb{N}$

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau$$
$$\leq 0 \quad (2.9)$$

Letting $n \to \infty$ in (2.9), we obtain that:

$$\lim_{n\to\infty} n d_n^k = 0 \quad (2.10)$$

From (2.10) $\exists\, n, \in \mathbb{N}$ such that

$n d_n^k \leq 1$ for all $n \geq n_1$

So we have,

$$d_n \leq \frac{1}{n^{\frac{1}{k}}} \quad (2.11)$$

for all $n \geq n_1$ In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.11) we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots \ldots \ldots + d(x_{m-1}, x_m)$$
$$\leq d_n + d_{n+1} + \cdots \ldots \ldots .. + d_{m-1}$$
$$= \sum_{i=n}^{m-1} d_i$$
$$\leq \sum_{i=n}^{\infty} d_i$$
$$\leq \sum_{i=n}^{\infty} 1/i\, 1/k \quad (2.12)$$

By convergence of the series $\sum_{i=1}^{\infty}\left(1/i^{\frac{1}{k}}\right)$ we get, $d(x_n, x_m) \to 0$ as $n \to \infty$. This yield that $\{x_n\}$ is a Cauchy sequence in (X, d) since (X, d) is complete metric space the sequence $\{x_n\}$ converges to some point $z \in X$; that is

$$\lim_{n\to\infty} x_n = z \quad (2.13)$$

Now suppose F is continues. In this case we claim that $z \in Tz$. Assume the contrary, that is $z \notin Tz$. In this case $\exists\, n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $D(x_{n_{k+1}}, Tz) > 0$ for all $n_k \geq n_0$. Otherwise $\exists\, n, \in \mathbb{N}$ such that $x_n \in Tz$ for all $n \geq n_1$ which implies that $z \in Tz$. This is a contradiction since $z \notin Tz$. Since $\delta(x_{n_{k+1}}, Tz) > 0$ for all $nk \geq n0$ then we have

$$\tau + F\left(D\left(x_{n_{k+1}}, Tz\right)\right) \leq \tau + F\left(\delta\left(x_{n_{k+1}}, Tz\right)\right)$$
$$\leq F\left(M_T\left(x_{n_k}, z\right)\right) \quad (2.14)$$

where

$M_T(x_{n_k}, z)$

$$= \max\left\{\begin{matrix} d(x_{n_k}, z), D(x_{n_k}, Tx_{n_k}), \\ D(z, Tz), \dfrac{D(z, Tz)[1 + D(x_{n_k}, Tx_{n_k})]}{1 + d(x_{n_k}, z)} \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} d(x_{n_k}, z), d(x_{n_k}, Tx_{n_k}), \\ d(z, Tz) \dfrac{d(z, Tz)[1 + d(x_{n_k}, Tx_{n_k})]}{1 + d(x_{n_k}, z)} \end{matrix}\right\}$$

Taking the limit $k \to \infty$ we have,

$$\lim_{k\to\infty} M_T(x_{n_k}, z) = d(z, Tz) \quad (2.15)$$

On letting $k \to \infty$ in (2.14) using (2.15) and continuity of F we have

$$\tau + F(D(z, Tz)) \leq F(D(z, Tz))$$

which is a contradiction. Thus we get $z \in Tz$. This completes the proof.

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