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Some double integrals involving g-function of two variables

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Abstract

Integrals are useful in connection with the study of certain boundary value problems. It is also helpful for obtaining the expansion formulae. It also used in the study of statistical distribution, probability and integral equation. Studying integrals is very important in our everyday life because we can use them to solve technical problems such as measuring the volume of water in a bucket. We can also use integrals to measure population density within a certain area. Integrals are vital in nearly every science and field of engineering. Anything that changes in time can be analyzed with integrals. In engineering, integrals are widely used to find applied forces, pressures and other phenomena that are dispersed over an area. The aim of this research paper is to evaluate some double integrals involving G-Function of two variables.

Key Words: *G-Function of two variables, Double Integral, Hypergeometric functions, Gamma Function.*

1. Introduction:

The G-function of two variables was defined by Shrivastava and Joshi [6, p. 471] in terms of Mellin-Barnes type integrals as follows:

$$G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}\\(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}\end{matrix}\right]$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\xi+\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\xi-\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\xi+\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\xi)}$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3}\Gamma(f_j-\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, $(i = 1, 2, 3; j = 2, 3)$.

The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \xi)$ $(j = 1, ..., m_2)$ lie to the right, and the poles of $\Gamma(1 - c_j + \xi)$ $(j = 1, ..., n_2)$, $\Gamma(1 - a_j + \xi + \eta)$ $(j = 1, ..., n_1)$ to the left of the contour.

The countor L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - \eta)$ $(j = 1, ..., m_3)$ lie to the right, and the poles of $\Gamma(1 - e_j + \eta)$ $(j = 1, ..., n_3)$, $\Gamma(1 - a_j + \xi + \eta)$ $(j = 1, ..., n_1)$ to the left of the contour, and the double integral converges if

$$2(n_1 + m_2 + n_2) > (p_1 + q_1 + p_2 + q_2)$$

$$2(n_1 + m_3 + n_3) > (p_1 + q_1 + p_3 + q_3)$$

and $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$.

where $U = [n_1 + m_2 + n_2 - \frac{1}{2}(p_1 + q_1 + p_2 + q_2)]$

$V = [n_1 + m_3 + n_3 - \frac{1}{2}(p_1 + q_1 + p_3 + q_3)]$

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These assumptions for the G-function of two variables will be adhered to throughout this research work.

The following formulae are required in the proof:

From Gradshteyn and Ryzik [3, p.372, (1) & (8)]:

$$\int_0^\pi (\sin x)^{\omega-1}\sin mx\,dx = \frac{\pi\sin\frac{m\pi}{2}\Gamma(\omega)}{2^{\omega-1}\Gamma\left(\frac{\omega+m+1}{2}\right)\Gamma\left(\frac{\omega-m+1}{2}\right)}, \tag{2}$$

$Re(\omega) > 0$.

$$\int_0^\pi (\sin x)^{\omega-1}\cos mx\,dx = \frac{\pi\cos\frac{m\pi}{2}\Gamma(\omega)}{2^{\omega-1}\Gamma\left(\frac{\omega+m+1}{2}\right)\Gamma\left(\frac{\omega-m+1}{2}\right)}, \tag{3}$$

$Re(\omega) > 0$.

From Erdelyi [2, p.4, (46)]:

The multiplication formula for the Gamma-Function

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2m}} m^{mz-\frac{1}{2}} \prod_{i=1}^{m-1}\Gamma\left(z+\frac{i}{m}\right), \tag{4}$$

where m is a positive integer.

2. INTEGRAL:

Certain double integrals involving the Generaliged Hypergeometric function and various polynomials have been evaluated by Kumari Shantha [4], Ayant Frédéric [1], Mishra Raghunayak [5] and others from time to time. Following them, we evaluate some double integrals involving the G-function of two variables.

In this section, we shall establish the following Integral:

$$\int_0^\pi\int_0^\pi (\sin x)^{\lambda-1}(\sin y)^{\mu-1}\sin rx\,\sin ty\; g(x,y)\,dx\,dy = \frac{\pi\sin\frac{r\pi}{2}\sin\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t); \tag{5}$$

$$\int_0^\pi\int_0^\pi (\sin x)^{\lambda-1}(\sin y)^{\mu-1}\sin rx\,\cos ty\; g(x,y)\,dx\,dy = \frac{\pi\sin\frac{r\pi}{2}\cos\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t); \tag{6}$$

$$\int_0^\pi\int_0^\pi (\sin x)^{\lambda-1}(\sin y)^{\mu-1}\cos rx\,\sin ty\; g(x,y)\,dx\,dy = \frac{\pi\cos\frac{r\pi}{2}\sin\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t); \tag{7}$$

$$\int_0^\pi\int_0^\pi (\sin x)^{\lambda-1}(\sin y)^{\mu-1}\cos rx\,\cos ty\; g(x,y)\,dx\,dy = \frac{\pi\cos\frac{r\pi}{2}\cos\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t); \tag{8}$$

where $2(m_3+n_3) > p_3+q_3, |\arg z_2| < \left(m_3+n_3-\frac{1}{2}p_3-\frac{1}{2}q_3\right)\pi$,

$Re\left(\lambda+2df_j\right) > 0, Re\left(\mu+2hf_j\right) > 0, j = 1,\ldots,m_3$;

and

$$g(x,y) = G^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix} z_1 \\ z_2(\sin x)^{2d}(\sin y)^{2h}\end{matrix}\middle|\begin{matrix} a_{p_1}:c_{p_2}:e_{p_3} \\ b_{q_1}:d_{q_2}:f_{q_3}\end{matrix}\right]$$

$$\psi(r,t) = G^{0,n_1;m_2,n_2;m_3,n_3+2d+2h}_{p_1,q_1;p_2,q_2;p_3+2d+2h,q_3+2d+2h}\left[\begin{matrix} z_1 \\ z_2(\sin x)^{2d}(\sin y)^{2h}\end{matrix}\middle|\begin{matrix} a_{p_1}:c_{p_2}:\Delta(2d,1-\lambda),\Delta(2h,1-\mu),e_{p_3} \\ b_{q_1}:d_{q_2}:f_{q_3},\Delta\left(d,\frac{1-\lambda\pm r}{2}\right),\Delta\left(h,\frac{1-\mu\pm t}{2}\right)\end{matrix}\right]$$

d and h are positive integers, the symbol $\Delta(d,\omega)$ represents the set of

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parameters $\frac{\omega}{d},\frac{\omega+1}{d},\ldots,\frac{\omega+d-1}{d}$ and the expression $\Delta\left(d,\frac{1-\omega\pm m}{2}\right)$ stands for $\Delta\left(d,\frac{1-\omega+m}{2}\right),\Delta\left(d,\frac{1-\omega-m}{2}\right)$.

Proof of (5):

To establish the integral (5), expressing the G-function in the integrand as the Mellin-Barnes type integral (1) and interchanging the orders of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,{z_1}^{\xi}{z_2}^{\eta}$$
$$\times\left(\int_0^{\pi}(sinx)^{\lambda+2d\eta-1}\,sinrx\,dx\right)$$
$$\times\left(\int_0^{\pi}(siny)^{\mu+2h\eta-1}\,sinty\,dy\right)d\xi d\eta$$

Evaluating the inner-integrals with the help of (2) and using the multiplication formula for gamma-function (4), we get

$$\frac{\pi sin\frac{r\pi}{2}sin\frac{t\pi}{2}}{\sqrt{(dh)}}\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,{z_1}^{\xi}{z_2}^{\eta}$$
$$\times\frac{\prod_{i=0}^{2d-1}\Gamma\left(\frac{\lambda+i}{2d}+\eta\right)}{\prod_{i=0}^{d-1}\Gamma\left(\frac{\frac{(\lambda+r+1)}{2}+i}{d}+\eta\right)}$$
$$\times\frac{\prod_{i=0}^{2h-1}\Gamma\left(\frac{\mu+i}{2h}+\eta\right)}{\prod_{i=0}^{d-1}\Gamma\left(\frac{\frac{(\lambda-r+1)}{2}}{d}+\eta\right)\prod_{i=0}^{h-1}\Gamma\left(\frac{\frac{(\mu+t+1)}{2}+i}{h}+\eta\right)\prod_{i=0}^{h-1}\Gamma\left(\frac{\frac{(\mu-t+1)}{2}+i}{h}+\eta\right)}d\xi d\eta$$

On applying (1), the value of the integral (5) is obtained.

On applying the same procedure as above and using (2) and (3), the integral (6) is established.

Similarly the integral (7) is established with the help of (3) and (2).

The integral (8) is established similarly with the help of (3).

3. Special cases:

On specializing the parameters in (5), (6), (7) and (8) we get following integral in terms of G-function of one variable:

$$\int_0^{\pi}\int_0^{\pi}(sinx)^{\lambda-1}(siny)^{\mu-1}sinrx\,sinty\,g(x,y)dxdy$$
$$=\frac{\pi sin\frac{r\pi}{2}sin\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t);\quad (9)$$

$$\int_0^{\pi}\int_0^{\pi}(sinx)^{\lambda-1}(siny)^{\mu-1}sinrx\,costy\,g(x,y)dxdy$$
$$=\frac{\pi sin\frac{r\pi}{2}cos\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t);\quad (10)$$

$$\int_0^{\pi}\int_0^{\pi}(sinx)^{\lambda-1}(siny)^{\mu-1}cosrx\,sinty\,g(x,y)dxdy$$
$$=\frac{\pi cos\frac{r\pi}{2}sin\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t);\quad (11)$$

$$\int_0^{\pi}\int_0^{\pi}(sinx)^{\lambda-1}(siny)^{\mu-1}cosrx\,costy\,g(x,y)dxdy$$
$$=\frac{\pi cos\frac{r\pi}{2}cos\frac{t\pi}{2}}{\sqrt{dh}}\psi(r,t);\quad (12)$$

where $2(m+n)>p+q,|argz|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,$

$Re\left(\lambda+2db_j\right)>0, Re\left(\mu+2hb_j\right)>0, j=1,\ldots,m;$

and

$g(x,y)=G_{p,q}^{m,n}\left[z(sinx)^{2d}(siny)^{2h}\middle|_{b_q}^{a_p}\right]$

$\psi(r,t)=G_{p+2d+2h,q+2d+2h}^{m,n+2d+2h}\left[z(sinx)^{2d}(siny)^{2h}\middle|_{b_q,\Delta\left(d,\frac{1-\lambda\pm r}{2}\right),\Delta\left(h,\frac{1-}{}\right.}^{\Delta(2d,1-\lambda),\Delta(2h,1-\mu)}\right.$

d and h are positive integers, the symbol $\Delta(d,\omega)$ represents the set of parameters $\frac{\omega}{d},\frac{\omega+1}{d},\ldots,\frac{\omega+d-1}{d}$ and the expression $\Delta\left(d,\frac{1-\omega\pm m}{2}\right)$ stands for $\Delta\left(d,\frac{1-\omega+m}{2}\right),\Delta\left(d,\frac{1-\omega-m}{2}\right)$.

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