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Some Vector Mixed Quasi-Complementarity Problems results

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Abstract:

I established the strong vector mixed quasi complementarity problems and the interrelated strong vector mixed quasi variational inequality problem. In **Banach spaces,** I demonstrated the equivalence between the strong mixed quasi complementarity problems and strong mixed quasi variational inequality problem. I used the KKM Fan lemma theorem and verified the existence of solutions of these problems, under pseudo monotonicity consideration. The facts displayed in this paper are addition and development of few previous and new facts in the literature.

1. Introduction

In 1980, Giannessi informed about the vector variational inequalities in a finite Dimensional Euclidean space. Chen and Cheng got inspired by Giannessi and studied the vector variational inequalities in infinite Dimensional Euclidean space and applied these to the vector optimization problems. From that time only, People have been studying the vector variational inequalities and their generalizations and applying it to vector optimization problems, vector complementarity problems and game theory. It is widely known to many of us that the complementarity problems are nearly related to variational inequality problems.Lemke and R.W. Cottle and G.B. Dantzig were the first one to interpolate the Complementarity theory.Many research scholars are widely exploring the applications of complementarity theory in their researches and coming forward with their hit and trials results which are being used in pure and applied sciences. With the scope of complementarity problems and its wide spreadness in different directions to study enormous problems arising in optimization,industries, physical, mathematical and engineering sciences. Beneath the pseudo monotone type conditions and positiveness type conditions, vector complementarity problems and their association with vector variational inequality problems have been explored.These results were studied,investigated and closely observed by renowned scholars, but only few current results on the strong version of the vector Variational inequality and vector complementarity problems were certified.

X.P. yang et al. was the one who analyzed the equivalence facts under some monotonicity constraints and some inclusive-type conditions in ordered Banach spaces among a vector complementarity problem, a vector variational inequality problem, a vector optimization problem, and weak minimal element problem.In 2005, N.J.huang and Fang proposed and interpolated several classes of strong vector F- Complementarity problems and gave still in existance of outcome for these problems in Banach spaces and considered the least element problems of feasible sets and demonstrated their relations with the strong vector F complementarity problems. S.A. khan interpolated and gave the analysis on the generalized vector implied Quasi Complementarity problems and generalized vector inferred Quasi Variational inequality problems. He determined the non emptiness and closedness of solution for the sets of these problems and verified that solution sets for both the problems are identical to each other with respect to some constraints.

I got inspired and motivated by the work going in these directions and thought of presenting the paper. In this paper, I interpolated and examined the new class of strong vector quasi complementarity problems and the corresponding strong vector mixed quasi variational inequality problems in the setting of Banach space and created the equivalence connection between these. By implementing the KKM Fan lemma,I proceeded with the existence of solution of strong vector mixed quasi variational inequality under pseudo monotonicity consideration and proved that the solution of the strong vector mixed quasi variational inequality is identical to the solution of strong vector mixed quasi complementarity problems under some conditions. The results provided in this paper are the

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generalizations and improvements done in already existing works under operation.

2. Preliminaries

Throughout this paper unless otherwise stated let X and Y be two real Banach spaces. Let $\mathcal{J}$ be a nonempty, closed, convex subset of a real Banach space X. A nonempty subset $Q \subset Y$ is called convex, pointed, connected, and reproduced cone, respectively, if it satisfies the following conditions:

(1) $\lambda Q \subseteq Q$,
for all $\lambda > 0$ and $Q+Q \subseteq Q$;
(2) $Q \cap -Q = \{0\}$;
(3) $Q \cup -Q = X$;
(4) $Q - Q = Q$.

Given Q in Y, we can define the relations "$\leq_Q$" and "$\nleq_Q$" as follows:

$$u \leq_Q v \Longleftrightarrow v-u \in Q,$$
$$u \nleq_Q v \Longleftrightarrow v-u \notin Q, \ \forall\, u, v \in Y. \qquad (1)$$

If "$\leq_Q$" is a partial order, then $(Y, \leq_Q)$ is called a Banach space ordered by Q. Let (X, Y) denote the space of all continuous linear mappings from X into Y.

Now, we rewrite some results and definitions useful for this paper.

Definition 1. A mapping $\mathcal{M} : \mathcal{J} \times \mathcal{J} \to Y$ is said to be Q-convex in 1st argument, if

$$\mathcal{M}(su + (1-s)v, w) \leq_Q s\,\mathcal{M}(u, w) + (1-s)\,\mathcal{M}(v, w), \quad \forall\ u, v, w \in \mathcal{J},\ 0 \leqslant s \leqslant 1 \qquad (2)$$

Definition 2. Let $\Im : \mathcal{J} \to L(X,Y)$ and $\mathcal{M} : \mathcal{J} \times \mathcal{J} \to Y$ be the two non-linear mappings. $\Im$ is said to be monotonic with respect to $\mathcal{M}$ if

$$\langle \Im u - \Im v, u - v\rangle + \mathcal{M}(v,u) - \mathcal{M}(u, u) \geq_Q 0, \quad \text{for all } u, v \in \mathcal{J}. \qquad (3)$$

Definition 3. Let : $\mathcal{J} \to (X,)$ and $\mathcal{M} : \mathcal{J} \times \mathcal{J} \to Y$ be the two non-linear mappings. $\Im$ is called to be pseudo monotone with respect to $\mathcal{M}$ if, for any given $u, v \in \mathcal{J}$,

$$\langle \Im u, v - u\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) \nleq_{Q\setminus\{0\}} 0$$
$$\Rightarrow \langle \Im v, -u\rangle + \mathcal{M}(v,u) - \mathcal{M}(u, u) \geq_Q 0. \qquad (4)$$

Remark 1. Every monotonic with respect to $\mathcal{M}$ is pseudo monotone with respect to $\mathcal{M}$ but converse of this not hold in general. Definition 3 is vector version of θ-pseudo monotonicity studied by A. khaliq, F.A khan.

Example 1. Let $X = \mathbb{R}$, $\mathcal{J} = \mathbb{R}+$, $Y = \mathbb{R}^2$, $Q = \mathbb{R}^2_+$, and

$$\Im(u) = \begin{bmatrix} 0 \\ \sin u \cos u \end{bmatrix}$$

$$\mathcal{M}(v,u) = \begin{bmatrix} v + u \\ v + u \end{bmatrix}, \text{ for all } u, v \in \mathcal{J}. \qquad (5)$$

Now, $\langle \Im u, v - u\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) =$

$$\begin{bmatrix} v - u \\ (1 + 2\sin u \cos u)(v - u) \end{bmatrix} \nleq_{Q\setminus\{0\}} 0. \qquad (6)$$

For $v \geq u$. It follows that

$$\langle \Im v, v - u\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) = \begin{bmatrix} v - u \\ (1 + 2\sin v \cos v)(v - u) \end{bmatrix} \geq_Q 0. \qquad (7)$$

So, $\Im$ is pseudo monotone with respect to $\mathcal{M}$. However, for $u = 2\pi$ and $v = 3\pi/2$, it follows that

$$\langle \Im u - \Im v, -v\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) = \begin{bmatrix} -\pi/2 \\ -\pi/2 \end{bmatrix} \geq Q0. \qquad (8)$$

From equation (8) we can say $\Im$ is not a monotonic with respect to $\mathcal{M}$.

Definition 4. A mapping $\Im : \mathcal{J} \to L(X,Y)$ is supposed to be hemi continuous if, for any $u, v \in \mathcal{J}$, the mapping $t \mapsto \langle \Im(u+s(v - u)), v - u\rangle$ has continuity at 0+.

Definition 5. A mapping $\mathcal{M} : \mathcal{J} \times \mathcal{J} \to Y$ is supposed to be positively homogeneous in 1st argument, if $\mathcal{M}(tu, v) = t\,\mathcal{M}(u, v)$ for all $u, v \in \mathcal{J}$ and $s \geq 0$.

Definition 6. Let $\mathcal{J}$ be a non-empty subset of a topological vector space U. A set-valued map $\Im : \mathcal{J}$

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$\rightarrow 2^u$ is supposed to be a KKM mapping if, for each non -empty finite subset$\{u_1, \ldots, u_n\} \subset \mathcal{J}$, co$\{u_1, \ldots, u_n\} \subset U_{i=1}^{n} \Im(u_i)$, where co used for convex hull.

Lemma 1 (KKM-Fan Lemma): Let $\mathcal{J}$ be a nonempty subset of Hausdorff topological vector *space* U. *Let* $\Im : \mathcal{J} \rightarrow 2^u$ *be a KKM-mapping such that for all* $u \in \mathcal{J}$, $\Im(u)$ *is closed and for at least one* $u \in \mathcal{J}$, $\Im(u)$ *is compact , then* $u \in \mathcal{J}$ *,*

$\cap \Im(u) \neq \Phi. \quad u \in \mathcal{J} \quad (9)$

3. Strong Vector Mixed Quasi Complementarity problems

All over this section, let X be a real Banach space and let $\mathcal{J} \subseteq X$ be a non -empty, closed, and convex subset of X. Let $(Y, \leq_Q)$ be an ordered Banach space induced by a pointed, closed, convex cone Q with non empty interior. Let $\mathcal{J} \rightarrow (X,Y)$ and $\mathcal{M} : \mathcal{J} \times \mathcal{J} \rightarrow Y$ be the two non linear mappings. In this paper, we consider the following strong vector mixed quasi complementarity problems:

(i) Strong vector mixed quasi complementarity problems $(SVMQCP)_1$:

Obtain $u \in \mathcal{J}$ for which $\langle \Im u, u\rangle + \mathcal{M}(u, u) \not\geq_{Q\setminus\{0\}} 0$, $\langle \Im u, v\rangle + \mathcal{M}(v, u) \nleq_{Q\setminus\{0\}} 0$, for all $v \in \mathcal{J}$.

(ii)Strong vector mixed Quasi complementarity problems $(SVMQCP)_2$:

Obtain $u \in \mathcal{J}$ for which$\langle \Im u, u\rangle + \mathcal{M}(u, u) = 0$, $\langle \Im u, v\rangle + \mathcal{M}(v, u) \nleq_{Q\setminus\{0\}} 0$, for all $v \in \mathcal{J}$.

Relatively to $(SVMQCP)_1$ and $(SVMQCP)_2$ problems, we assume that the following strong vector mixed quasi variational inequality problems:

(SVMQVIP):Obtain $u \in \mathcal{J}$ for which $\langle \Im u, v - u\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) \nleq_{Q\setminus\{0\}} 0$, for all $v \in \mathcal{J}$.

The strong vector mixed quasi variational inequality problem (SVMQVIP) is the generalization and extension of some already known vectors along with scalar mixed quasi variational inequalities. For the formulation, numerical results, existence results, sensitivity analysis, and dynamical point of view the mixed quasi variational inequalities and references there in.

Remark 2. (a) If $Y = \mathbb{R}$ and $Q = \mathbb{R}+$, then $(SVMQCP)_1$ and $(SVMQCP)_2$ and (SVMQVIP) reduced, correspondingly, to the mixed quasi complementarity problems (MQCP):(MQCP).Obtain $u \in \mathcal{J}$ for which $\langle \Im u, v-u\rangle + \mathcal{M}(u, u) = 0$, $\langle \Im u, v\rangle + \mathcal{M}(v, u) \geq 0$, for all $v \in \mathcal{J}$

and mixed quasi variational inequality problem (MQVIP):(MQVIP)

Find u ∈ for which $\langle \Im u, v-u\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) \geq 0$, for all $v \in \mathcal{J}$,

that was introduced and studied by M.aslamnoor et al.

(b) If $\mathcal{M} = 0$, thus $(S\,V\,M\,Q\,C\,P)_1$ and $(S\,V\,M\,Q\,C\,P)_2$ reduced to the following strong vector complementarity problems (S V C P):

$(S\,V\,C\,P)_1$ obtain $u \in \mathcal{J}$ for which $\langle \Im u, u\rangle \not\geq_{Q\setminus\{0\}} 0$, $\langle \Im u, v\rangle \nleq_{Q\setminus\{0\}} 0$, for all $v \in \mathcal{J}$,

$(S\,V\,C\,P)_2$ obtain $u \in \mathcal{J}$ for which $\langle \Im u, u\rangle = 0$, $\langle \Im u, v\rangle \nleq_{Q\setminus\{0\}} 0$, for all $v \in \mathcal{J}$.

And (S V M Q V I P) reduces to the following strong vector variational inequality problems (SVVIP):

(S V V I P) obtain $u \in K$ for which $\langle \Im u, v - u\rangle \nleq_{Q\setminus\{0\}} 0$, $\forall v \in \mathcal{J}$.

Firstly, we will examine the equivalences among $(S\,V\,M\,Q\,C\,P)_1$ and $(S\,V\,M\,Q\,C\,P)_2$ and (S V M Q V I P), under some applicable considerations .

Theorem 1. (i) Consider that $\langle \Im w, w\rangle + \mathcal{M}(w, w) \in Q \cup (-Q)$, *for all* $w \in \mathcal{J}$. If u solves $(SVMQCP)_1$ *then* u solves *(SVMQVIP)*

(ii) *Let* $\mathcal{M} : \mathcal{J} \times \mathcal{J} \rightarrow Y$*satisfy* $\mathcal{M}(2u, v) = 2\mathcal{M}(u, v)$, $\forall\ u, v \in \mathcal{J}$ *and* $\langle \Im w, w\rangle + \mathcal{M}(w, w) \in Q \cup (-Q)$, $\forall w \in \mathcal{J}$. If u *solves (SVMQVIP) therefore* u *also solves* $(SVMQCP)_1$

Proof: (i) Let $u \in \mathcal{J}$ be the solution of $(SVMQCP)_1$. Then $u \in \mathcal{J}$ such that

$$\langle \Im u, u\rangle + \mathcal{M}(u, u) \not\geq_{Q\setminus\{0\}} 0, \quad (1)$$

$$\langle \Im u, v\rangle + \mathcal{M}(v, u) \nleq_{Q\setminus\{0\}} 0, \quad \forall v \in \mathcal{J}. \quad (2)$$

Putting $v = u$ in (11), we obtain

$$\langle \Im u, u\rangle + \mathcal{M}(u, u) \nleq_{Q\setminus\{0\}} 0. \quad (3)$$

Due to $\langle \Im w, w\rangle + \mathcal{M}(w, w) \in Q \cup (-Q)$, for all $w \in \mathcal{J}$, we have

$$\langle \Im u, u\rangle + \mathcal{M}(u, u) \geq_Q 0 \text{ or } \langle \Im u, u\rangle + \mathcal{M}(u, u) \leq_Q 0. \quad (4)$$

From (1), (3), and (4), we have

$$\langle \Im u, u\rangle + \mathcal{M}(u, u) = 0. \quad (5)$$

From (2) and (5), we have

$$\langle \Im u, v - u\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) = \langle \Im u, v\rangle + \mathcal{M}(v, u) - \langle \Im u, u\rangle - \mathcal{M}(u, u) = \langle \Im u, v\rangle + \mathcal{M}(v, u) \nleq_{Q\setminus\{0\}} 0, \quad (6)$$

$\forall v \in \mathcal{J}$. Hence , $u \in \mathcal{J}$ is the solution of (SVMQVIP).

(ii) Now, let $u \in \mathcal{J}$ be the solution of (SVMQVIP), therefore

$$\langle \Im u, v - u\rangle + \mathcal{M}(v, u) - \mathcal{M}(u, u) \nleq_{Q\setminus\{0\}} 0, \forall v \in \mathcal{J}. \quad (1)$$

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Since $\mathscr{M}(2u, v) = 2\mathscr{M}(u, v)$, for all $u, v \in \mathcal{J}$, then it follows that $\mathscr{M}(0, v) = 0$, for all $v \in \mathcal{J}$. By putting $v = 2u$ and $v = 0$, respectively, in (16), we get

$$\langle \Im u, u\rangle + \mathscr{M}(u, u) \nleq_{Q\setminus\{0\}} 0,$$
$$\langle \Im u, u\rangle + \mathscr{M}(u, u) \ngeq_{Q\setminus\{0\}} 0. \quad (2)$$

Since $\langle \Im w, w\rangle + \mathscr{M}(w, w) \in \mathcal{Q} \cup (-\mathcal{Q})$, $\forall\ w \in K$, we have

$$\langle \Im u, u\rangle + \mathscr{M}(u, u) \geq_Q 0 \ \text{ or } \ \langle \Im u, u\rangle + \mathscr{M}(u, u) \leq_Q 0. \quad (3)$$

From (2) and (3), we have

$$\langle \Im u, u\rangle + \mathscr{M}(u, u) = 0. \quad (4)$$

Now, from Inclusions (1) and (4), we have

$$\langle \Im u, v\rangle + \mathscr{M}(v, u) = \langle \Im u, v - u\rangle + \mathscr{M}(v,u) - \mathscr{M}(u, u) + \langle \Im u, u\rangle + \mathscr{M}(u, u)$$
$$= \langle \Im u, v - u\rangle + \mathscr{M}(v,u) - \mathscr{M}(u, u)$$
$$\nleq_{Q\setminus\{0\}} 0, \quad (5)$$

$\forall\ v \in \mathcal{J}$, this implies that u solves $(SVMQCP)_1$.

Remark 3. The condition $\mathscr{M}(2u, v) = 2\mathscr{M}(u, v)$, $\forall\ u, v \in \mathcal{J}$ holds if $\mathscr{M}$ is positively Homogeneous; i.e., $\mathscr{M}(su, v) = s\mathscr{M}(u, v)$ for all $s \geq 0$. Hence, Theorem 1 generalizes and improves the theorems in [6, 9, 11, 14, 15]. Here we give an example of a function $\mathscr{M}$, which fulfilled the condition $\mathscr{M}(2u, v) = 2\mathscr{M}(u, v)$, for all $u, v \in \mathcal{J}$ but not a positively Homogeneous,this implies that already known results in [6, 9, 11, 14, 15] cannot be applied.

Example 2. Let $\mathscr{M} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$\mathscr{M}(u,y) = \begin{cases} 2u, & \text{if } u \in Q, \\ 0, & \text{if } u \in Q^c. \end{cases} \quad (1)$$

Thus $\mathscr{M}$ fulfill the condition $\mathscr{M}(2u, v) = 2\mathscr{M}(u, v)$ but it is not positively homogeneous.

Theorem 2. (a) If u solves problem $(S\,V\,M\,Q\,C\,P)_2$ then u solves (S V M Q V I P).

(b) Let $\mathscr{M} : \mathcal{J} \times \mathcal{J} \to Y$ and $\mathscr{M}(2u, v) = 2\mathscr{M}(u, v)$, for all $u, v \in \mathcal{J}$ and $\langle \Im w, w\rangle + \mathscr{M}(w, w) \in \mathcal{Q} \cup (-\mathcal{Q})$, for all $w \in \mathcal{J}$. If u solves (SVMQVIP) then u solves $(SVMQCP)_2$.

Proof:(a) Let $u \in K$ be the solution of $(SVMQCP)_2$. Then $u \in K$ such that

$$\langle \Im u, u\rangle + \mathscr{M}(u, u) = 0,$$
$$\langle \Im u, v\rangle + \mathscr{M}(v, u) \nleq_{Q\setminus\{0\}} 0, \ \forall\ v \in \mathcal{J}. \quad (1)$$

Now,

$$\langle \Im u, v - u\rangle + \mathscr{M}(v, u) - \mathscr{M}(u, u) = \langle \Im u, v\rangle + \mathscr{M}(v,u) - \langle \Im u, u\rangle - \mathscr{M}(u, u)$$
$$= \langle \Im u, v\rangle + \mathscr{M}(v,u)$$
$$\nleq_{Q\setminus\{0\}} 0, \quad (2)$$

for all $v \in \mathcal{J}$. Hence , $u \in \mathcal{J}$ is the solution of (S V M Q V I P).

(b) Now, let $u \in$ be the solution of (S V M Q V I P), so

$$\langle \Im u, v - u\rangle + \mathscr{M}(v, u) - \mathscr{M}(u, u) \nleq_{Q\setminus\{0\}} 0, \ \forall v \in \mathcal{J}. \quad (1)$$

As $\mathscr{M}(2u, v) = 2\mathscr{M}(u, v)$, for all $u, v \in \mathcal{J}$, therefore it follows that $\mathscr{M}(0, v) = 0$, for all $v \in \mathcal{J}$. By putting $v = 2u$ and $v = 0$, correspondingly, in (24), we get

$$\langle \Im u, u\rangle + \mathscr{M}(u, u) \nleq_{Q\setminus\{0\}} 0,$$
$$\langle \Im u, u\rangle + \mathscr{M}(u, u) \ngeq_{Q\setminus\{0\}} 0. \quad (2)$$

Since $\langle \Im w, w\rangle + \mathscr{M}(w, w) \in \mathcal{Q} \cup (-\mathcal{Q})$, for all $w \in \mathcal{J}$, we have

$$\langle \Im u, u\rangle + \mathscr{M}(u, u) \geq_Q 0 \text{ or } \langle \Im u, u\rangle + \mathscr{M}(u, u) \leq_Q 0. \quad (3)$$

From inclusions (2) and (3), we get

$$\langle \Im u, u\rangle + \mathscr{M}(u, u) = 0. \quad (4)$$

By use of (4), we have

$$\langle \Im u, v\rangle + \mathscr{M}(v, u) = \langle \Im u, v - u\rangle + \mathscr{M}(v,u) - \mathscr{M}(u, u) + \langle \Im u, u\rangle + \mathscr{M}(u, u)$$
$$= \langle \Im u, v - u\rangle + \mathscr{M}(v,u) - \mathscr{M}(u, u)$$
$$\nleq_{Q\setminus\{0\}} 0, \quad (5)$$

for all $v \in \mathcal{J}$. Then (4) and (5) implies that u solves $(SVMQCP)_2$.

4. Existing Results

Firstly, we will prove some Minty-type lemma with the support of pseudo monotone mapping with respect to $\mathscr{M}$.

Lemma 2. Let $\mathscr{M} : \mathcal{J} \times \mathcal{J} \to Y$ be $\mathcal{Q}$-convex in 1st argument and let $\Im : \mathcal{J} \to L(X,Y)$ be a hemi continuous mapping and pseudo monotone with respect to $\mathscr{M}$. Thus the following two problems are equivalent:

(a) $u \in \mathcal{J}$, $\langle \Im u, v - u\rangle + \mathscr{M}(v, u) - \mathscr{M}(u, u) \nleq_{Q\setminus\{0\}}$, $\forall v \in \mathcal{J}$, (1)

(b) $u \in \mathcal{J}$, $\langle \Im v, v - u\rangle + \mathscr{M}(v, u) - \mathscr{M}(u, u) \geq_Q 0$, $\forall v \in \mathcal{J}$. (2)

Proof: (1) ⇒ (2). The result directly follows from pseudo monotonicity with respect to $\mathscr{M}$.

Now, (2) ⇒ (1). For any given $v \in \mathcal{J}$, we know that $v_s = sv + (1 - s)u \in \mathcal{J}$, for all $s \in (0, 1)$, since $\mathcal{J}$ is convex. Since $u \in \mathcal{J}$ is a solution of problem (2), so for each $u \in \mathcal{J}$, it

Follows that $\langle \Im v_s, v_s - u\rangle + \mathscr{M}(v_s, u) - \mathscr{M}(u, u) \geq_Q 0$. (3)

Now, we have $s\langle \Im y_s, - u\rangle + s(\mathscr{M}(v,u) - \mathscr{M}(u, u)) \geq \langle \Im v_s, v_s - u\rangle + \mathscr{M}(v_s, u) - \mathscr{M}(u, u) \geq_Q 0$. (4)

For $s > 0$, we have

$$\langle \Im y_s, v - u\rangle + \mathscr{M}(v,u) - \mathscr{M}(u, u) \geq_Q 0. \quad (5)$$

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Since ℑ is hemi continuous and Q is closed, letting s → 0+ in (5), we get

⟨ ℑ u, v − u⟩ + $\mathscr{M}$ (v, u) − $\mathscr{M}$ (u, u) $\geq_Q$ 0, ∀v ∈ $\mathcal{J}$. (6)

Hence,

⟨ ℑ u, v − u⟩ + $\mathscr{M}$ (v, u) − $\mathscr{M}$ (u, u) $\nleq_{Q\setminus\{0\}}$, ∀v∈ $\mathcal{J}$. (7)

thus, u ∈ $\mathcal{J}$ is solution of problem (1). This proves the result.

Theorem 3. Let X be real reflexive Banach space and let Y be a Banach space. Let $\mathscr{M} \subset X$ be a nonempty, bounded, closed, and convex subset. Let $\mathscr{M}$: $\mathcal{J} \times \mathcal{J} \rightarrow Y$ be Q-convex and upper semi continuous in first and second arguments, respectively . Let ℑ : $\mathcal{J} \rightarrow L(X,Y)$ be hemi continuous and pseudo monotone with respect to $\mathscr{M}$. Thus (S V M Q V I P) has solution.

Proof: Consider two set valued mappings $\mathscr{M}$: $K \rightarrow 2^K$ as follows:

G(v) = {u ∈ : ⟨ ℑ u,v − u⟩ + $\mathscr{M}$ (v, u) − $\mathscr{M}$ (u, u) $\nleq_{Q\setminus\{0\}}$} , ∀v ∈ $\mathcal{J}$,

H(v) = {u ∈ : ⟨ ℑ v,v − u⟩+ $\mathscr{M}$ (v, u) − $\mathscr{M}$ (u, u) $\geq_Q$0} , ∀v ∈ $\mathcal{J}$. (1)

G(v) and H (v) are non empty, as v ∈ G(v) ∩ H(v). We claim that G is a KKM mapping. If this is not true, then there exist a finite set {v1, . . . , vn} ⊂ K and si≥ 0, i = 1, . . . , n with $\Sigma_{i=1}^{n} s_i = 1$ such that v = $\Sigma_{i=1}^{n} s_i v_i \notin \bigcup_{i=1}^{n} G(v_i)$. Now, by the definition of G, we have

⟨ ℑ v, vi− v⟩ + $\mathscr{M}$ (vi, v) − $\mathscr{M}$ (v, v) $\leq_{Q\setminus\{0\}}$,i = 1, . . . , n. (2)

Now, we have

0 = ⟨ ℑ v, v − v⟩ + $\mathscr{M}$ (v, v) − $\mathscr{M}$ (v, v)

= ⟨ ℑ v,$\Sigma_{i=1}^{n} s_i v_i$− v⟩+ $\mathscr{M}$ ($\Sigma_{i=1}^{n} s_i v_i$, v) − $\mathscr{M}$ (v, v)

=$\Sigma_{i=1}^{n}$[⟨ℑ v, vi− v⟩ + $\mathscr{M}$ (v i, v) − $\mathscr{M}$ (v, v)]

$\leq_{Q\setminus\{0\}}$, (3)

which is not possible. Thus, our claim is verified. So is KKM mapping.

Now, since ℑ is pseudo monotone with respect to $\mathscr{M}$, thus G (v) ⊂H (v) for every v ∈ $\mathcal{J}$ and so H is also a KKM mapping. Now we claim that for each v ∈ ,H (v) ⊂ $\mathcal{J}$ is closed in the weak topology of X.

Indeed, consider x ∈ $\overline{(v)}^w$, the weak closure of H(v).

Since X is reflexive, there is a sequence {u_n} in H(v) s.t. {u_n} converges weakly to u ∈ $\mathcal{J}$. Then

⟨ ℑ v, v – u_n⟩ + $\mathscr{M}$ (v,u_n) − $\mathscr{M}$ (u,u_n) $\geq_Q$ 0. (4)

Since $\mathscr{M}$ (v, ·) is upper semi continuous and Q is closed, therefore,

⟨ ℑ v, v − u⟩ + $\mathscr{M}$ (v, u) − $\mathscr{M}$ (u, u) $\geq_Q$0 (5)

and so u ∈ $\overline{H}$(v). This shows that H (v) is weakly closed, for each v ∈ $\mathcal{J}$. Our claim is then clearly verified. Since X is reflexive and $\mathcal{J} \subset X$ is non empty, bounded, closed and convex, $\mathcal{J}$ is a weakly compact subset of X and so (v) is also weakly compact. According to Lemma 1 (KKM-Fan Lemma),

$\bigcap_{v\in K}$ H (v) ≠ Φ. (6)

This implies that there exists u ∈ for which

⟨ ℑ v, v − u⟩ + $\mathscr{M}$ (v, u) − $\mathscr{M}$ (u, u) $\geq_Q$0, ∀v ∈ $\mathcal{J}$. (7)

Therefore by Lemma 2, we observe that there exists u ∈ $\mathcal{J}$ such that

⟨ ℑ u, v − u⟩ + $\mathscr{M}$ (v, u) − $\mathscr{M}$ (u, u) $\nleq_{Q\setminus\{0\}}$, ∀ v∈ $\mathcal{J}$. (8)

Thus our result is proved.

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